Problem 3.1 on Page 212

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1 The Problem

On page 212 of NFCM [1], we find problem (3.1): Equation (3.6) [pg. 207] shows that the orbital distance $r = r(\hat{\mathbf{r}})$ has either minimum or maximum values $r_\pm = |\,\mathbf{r}_\pm\,| = r(\pm\epsilon)$ when $\hat{\mathbf{r}} = \pm\hat{\boldsymbol{\epsilon}}$. The *semi-major axis* a is defined as half the distance between the points $\mathbf{r}_+$ and $\mathbf{r}_-$. The *semi-latus rectum* ℓ is defined as the orbital distance when $\mathbf{r}\cdot\boldsymbol{\epsilon} = 0$ in the attractive case or $\mathbf{r}\cdot\boldsymbol{\epsilon} = -2$ in the repulsive case. Verify the following relations:
[I will probably will not do them all.]

For an ellipse

$$r_\pm = \frac{\ell}{1 \pm \epsilon} = a(1 \mp \epsilon) \quad \text{where} \quad a \equiv \tfrac{1}{2}(r_+ + r_-)\,, \tag{1}$$

$$\frac{1}{\ell} = \frac{1}{2}\Big(\frac{1}{r_+} + \frac{1}{r_-}\Big)\,, \tag{2}$$

$$\epsilon = \frac{r_+ - r_-}{r_+ + r_-}\,, \tag{3}$$

$$\frac{v_+}{v_-} = \frac{1+\epsilon}{1-\epsilon}\,, \tag{4}$$

where $v_\pm \equiv |\,\mathbf{v}(\pm\hat{\boldsymbol{\epsilon}})\,|$.

Problem 1: From (1),

$$r_+ = \frac{\ell}{1+\epsilon}\,, \quad r_- = \frac{\ell}{1-\epsilon}\,, \tag{5}$$

Then,

$$2a = r_+ + r_- = \frac{\ell}{1+\epsilon} + \frac{\ell}{1-\epsilon}\,, \tag{6}$$

hence

$$a = r_+ + r_- = \frac{\ell}{(1+\epsilon)(1-\epsilon)}\,. \tag{7}$$

Therefore,

$$\frac{r_+}{a} = 1 - \epsilon \quad \text{and} \quad \frac{r_-}{a} = 1 + \epsilon\,. \tag{8}$$

And finally,

$$r_\pm = a(1 \mp \epsilon)\,. \tag{9}$$

Problem 2: From (5),

$$\frac{1}{2}\Big(\frac{1}{r_+}+\frac{1}{r_-}\Big)=\frac{1}{2}\Big(\frac{1+\epsilon}{\ell}+\frac{1-\epsilon}{\ell}\Big)=\frac{1}{\ell}\,, \tag{10}$$

Problem 3: From (3),

$$\frac{r_+-r_-}{r_++r_-}=\frac{(1+\epsilon)-(1-\epsilon)}{(1+\epsilon)+(1-\epsilon)}=\epsilon\,. \tag{11}$$

Problem 4: Okay, I have to interpret what this all means. I interpret v_+ as the speed of the particle as it traverses the major axis on the $+\boldsymbol{\epsilon}$ side (so that $\mathbf{v}_+\cdot\hat{\mathbf{r}}=0$, $\mathbf{v}_+\wedge\hat{\mathbf{r}}=v_+\mathbf{i}^\dagger$, and $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\epsilon}}=+1$), and, likewise, v_- as the speed of the particle as it traverses the major axis on the $-\boldsymbol{\epsilon}$ side (so that $\mathbf{v}_-\cdot\hat{\mathbf{r}}=0$, $\mathbf{v}_-\wedge\hat{\mathbf{r}}=-v_-\mathbf{i}=v_-\mathbf{i}^\dagger$, and $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\epsilon}}=-1$).

Now, from Eq. (3.7) on page 207, we have

$$\mathbf{L}\mathbf{v}=k(\boldsymbol{\epsilon}+\hat{\mathbf{r}})\,. \tag{12}$$

If we multiply both sides of this on the right by $\hat{\mathbf{r}}$, we get

$$\mathbf{L}\mathbf{v}\hat{\mathbf{r}}=k(\boldsymbol{\epsilon}\hat{\mathbf{r}}+1)\,. \tag{13}$$

This last equation is still general. Now, we'll use it twice, once when $\mathbf{v}=\mathbf{v}_+$ and once when $\mathbf{v}=\mathbf{v}_-$. From this we get that

$$\mathbf{L}\mathbf{v}_+\hat{\mathbf{r}}=k(\epsilon+1)\,, \tag{14a}$$
$$\mathbf{L}\mathbf{v}_-\hat{\mathbf{r}}=k(-\epsilon+1)\,, \tag{14b}$$

which gives

$$\mathbf{L}v_+\mathbf{i}^\dagger=k(\epsilon+1)\,, \tag{15a}$$
$$\mathbf{L}v_-\mathbf{i}^\dagger=k(-\epsilon+1)\,. \tag{15b}$$

Cross multiplying these last two equations yields

$$k(-\epsilon+1)v_+\mathbf{L}\mathbf{i}^\dagger=k(\epsilon+1)v_-\mathbf{L}\mathbf{i}^\dagger\,. \tag{16}$$

On simplifying, we get

$$(-\epsilon+1)v_+=(\epsilon+1)v_-\,, \tag{17}$$

which gives us the proportionality

$$\frac{v_+}{v_-}=\frac{1+\epsilon}{1-\epsilon}\,. \tag{18}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.