Problem 3.4 on Page 214

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1 The Problem

On page 214 of NFCM [1], we find problem (3.4): Show that orbital distance can be expressed as a function of velocity by

$$r = r(\mathbf{v}) = \frac{-k}{2E - m\mathbf{u}\cdot\mathbf{v}}\,, \tag{1}$$

where

$$\mathbf{u} \equiv k\mathbf{L}^{-1}\boldsymbol{\epsilon}\,, \tag{2}$$

which is Eq. (3.8) in the text. For later use, we also get that

$$\mathbf{u}^2 = \frac{k^2\epsilon^2}{L^2}\,. \tag{3}$$

2 The Solution

[Partial answer]

From (3.5) we have (for later use)

$$\epsilon^2 - 1 = \frac{2L^2E}{mk^2}\,. \tag{4}$$

Equation (3.9) is

$$(\mathbf{v}-\mathbf{u})^2 = \frac{k^2}{L^2}\,. \tag{5}$$

Expanding this, we get

$$\mathbf{v}^2 - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{u}^2 = \frac{k^2}{L^2}\,. \tag{6}$$

Substituting from (3) into this and rearranging, we have

$$\mathbf{v}^2 - 2\mathbf{u}\cdot\mathbf{v} = -\frac{k^2}{L^2}(\epsilon^2 - 1)\,. \tag{7}$$

Using Eq. (4) and then simplifying gives us

$$\tfrac{1}{2}m\mathbf{v}^2 - m\mathbf{u}\cdot\mathbf{v} = -E\,. \tag{8}$$

Now, we can get an equation for r from the basic energy equation

$$E = \tfrac{1}{2}m\mathbf{v}^2 - \frac{k}{r}\,. \tag{9}$$

On solving this for r, we get

$$r = \frac{-k}{E - \frac{1}{2}m\mathbf{v}^2}\,, \tag{10}$$

Substituting in from (9), we get

$$r = r(\mathbf{v}) = \frac{-k}{2E - m\mathbf{u}\cdot\mathbf{v}}\,, \tag{11}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.