Notes on Chapter 5, Section 1

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1 Introduction

These notes cover pages 252 to 261 of NFCM [1].

2 Linear Functions and Matrices: Adjoints and Inverses

We begin with the projection operator.

$$P_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}^{-1}\mathbf{a}\cdot\mathbf{x}\,. \tag{1}$$

But since

$$\mathbf{a}\cdot\mathbf{x} = \tfrac{1}{2}(\mathbf{ax}+\mathbf{xa})\,, \tag{2}$$

then

$$P_{\mathbf{a}}(\mathbf{x}) = \tfrac{1}{2}\mathbf{a}^{-1}(\mathbf{ax}+\mathbf{xa}) = \tfrac{1}{2}(\mathbf{x}+\mathbf{a}^{-1}\mathbf{xa})\,. \tag{3}$$

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The text says that for every linear function f that takes values in the range space and an arbitrary vector $\mathbf{y}$ in the range space, there exists a linear function from the range space back to the domain space such that the followng relation holds true:

$$\bar{f}(\mathbf{y})\cdot\mathbf{x} = \mathbf{y}\cdot f(\mathbf{x})\,, \tag{4}$$

where $\bar{f}$ is called the *adjoint* of f. Just looking at the LHS side of this last equation, it seems obvious that $\bar{f}$ takes it values in the domain space because it's dotting $\mathbf{x}$ which is a domain vector. However, the implicit deinfition of a function does not guarantee its existence. But we can do that by solving for $\bar{f}$, from which we get that

$$\bar{f}(\mathbf{y}) = \mathbf{y}\cdot\nabla_{\mathbf{x}}f(\mathbf{x})\,. \tag{5}$$

To prove this, we just multiply through both sides of (4) by $\nabla_{\mathbf{x}}$ to get

$$\nabla_{\mathbf{x}}\bar{f}(\mathbf{y})\cdot\mathbf{x} = \nabla_{\mathbf{x}}(\mathbf{y}\cdot f(\mathbf{x}))\,. \tag{6}$$

We'll manipulate the left-hand side and right-hand side separately. In both cases we use the fact that $\mathbf{y}$ is not a function of $\mathbf{x}$. An indentity we'll need is

$$\partial_i x_j = \frac{\partial x_j}{\partial x_i} = \delta_{ij}\,. \tag{7}$$

Starting with the LHS:

$$\begin{aligned}\nabla_{\mathbf{x}}\bar{f}(\mathbf{y})\cdot\mathbf{x} &= \nabla_{\mathbf{x}}[\bar{f}(\mathbf{y})]_j x_j \\ &= \sigma_i\partial_i[\bar{f}(\mathbf{y})]_j x_j \\ &= \sigma_i\delta_{ij}[\bar{f}(\mathbf{y})]_j \\ &= \sigma_j[\bar{f}(\mathbf{y})]_j \\ &= \bar{f}(\mathbf{y})\,. \end{aligned} \tag{8}$$

Now for the right-hand side:

$$\begin{aligned}\nabla_{\mathbf{x}}(\mathbf{y}\cdot f(\mathbf{x})) &= \nabla_{\mathbf{x}}(\sigma_k y_k \cdot x_j f(\sigma_j)) \\ &= \nabla_{\mathbf{x}} y_k x_j(\sigma_k\cdot f(\sigma_j)) \\ &= \sigma_i\partial_i y_k x_j f_{jk} \\ &= \sigma_i\delta_{ij} y_k f_{jk} \\ &= \sigma_j y_k f_{jk}\,. \end{aligned} \tag{9}$$

And we're halfway there.

$$\begin{aligned}\mathbf{y}\cdot\nabla_{\mathbf{x}} f(\mathbf{x})) &= y_i\partial_i(x_j f(\sigma_j)) \\ &= y_i\delta_{ij} f(\sigma_j) \\ &= y_i f(\sigma_i) \\ &= y_i f_{ji}\sigma_i \\ &= \sigma_j y_k f_{jk}\,. \end{aligned} \tag{10}$$

Hence,

$$\bar{f}(\mathbf{y}) = \nabla_{\mathbf{x}}\bar{f}(\mathbf{y})\cdot\mathbf{x} = \nabla_{\mathbf{x}}(\mathbf{y}\cdot f(\mathbf{x})) = \mathbf{y}\cdot\nabla_{\mathbf{x}} f(\mathbf{x}))\,. \tag{11}$$

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Let f be a linear function from the domain of 3-vectors to the range or 3-vectors. We let $\underline{f}$ be the extension of f to act linearly on all n-graded objects, from scalars to vector to bivectors to pseudoscalars. In particular, for any three domain vectors, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$,

$$\underline{f}(\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}) \equiv f(\mathbf{x})\wedge f(\mathbf{y})\wedge f(\mathbf{z})\,. \tag{12}$$

Generally,

$$\underline{f}(\mathbf{x}_1\wedge\mathbf{x}_2\wedge\cdots\wedge\mathbf{x}_k) \equiv f(\mathbf{x}_1)\wedge f(\mathbf{x}_2)\wedge\cdots\wedge f(\mathbf{x}_k)\,. \tag{13}$$

This preseravtion of the number of wedges of the domain being preserved in the range is called an *outermorphism*. Since $f(\mathbf{x})\wedge f(\mathbf{y})\wedge f(\mathbf{z})$ must be proportional to the original pseudoscalar $\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}$ then we can write

$$\underline{f}(\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}) = (\det f)\,\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}\,, \tag{14}$$

where $\det f$ is the scalar function of proportionality. A canonical way to solve for $\underline{f}$ is this

$$\underline{f}(i) = (\det f)\,i\,, \tag{15}$$

from which we get

$$\det f = i^{-1}\underline{f}(i) = i^{-1}\bar{f}(i)\,. \tag{16}$$

The proof that $\underline{f}(i) = \bar{f}(i)$ is given below.

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We now have arrived at what I consider the main result of this section: The inverse of a linear transformation.

Definiiton: If f is a linear transformation, then f is said to be *singular* if $\det f = 0$, and *nonsingular*, otherwise.

Main Result:

Assuming that f is a linear, nonsingular transformation, then

$$f^{-1}(\mathbf{y}) = \bar{f}(\mathbf{y}i)/\bar{f}(i) = \frac{\bar{f}(\mathbf{y}i)i^{-1}}{\det f}\,, \tag{17}$$

Proof:

Given: f is a linear, nonsingular transformation, such that

$$f(\mathbf{x}) = \mathbf{y}\,. \tag{18}$$

Now, consider

$$\mathbf{x}\bar{f}(i) = \mathbf{x}\bar{f}(\sigma_1 \wedge \sigma_2 \wedge \sigma_3) = \mathbf{x}\cdot\bar{f}(\sigma_1)\wedge\bar{f}(\sigma_2)\wedge\bar{f}(\sigma_3)\,, \tag{19}$$

where we have used two simple facts. First, that $\mathbf{x}\wedge\bar{f}(i) \equiv 0$ and, second, that $\bar{f}$ is itself an outermorphism. Continuing, we have that

$$\mathbf{x}\bar{f}(i) = \mathbf{x}\cdot\bar{f}(\sigma_1)\,\bar{f}(\sigma_2)\wedge\bar{f}(\sigma_3) - \mathbf{x}\cdot\bar{f}(\sigma_2)\,\bar{f}(\sigma_1)\wedge\bar{f}(\sigma_3) + \mathbf{x}\cdot\bar{f}(\sigma_3)\,\bar{f}(\sigma_1)\wedge\bar{f}(\sigma_2)\,. \tag{20}$$

But we know that[1]

$$\mathbf{x}\cdot\bar{f}(\sigma_k) = \underline{f}(\mathbf{x})\cdot\sigma_k = f(\mathbf{x})\cdot\sigma_k\,. \tag{21}$$

Hence, (20) becomes

$$\mathbf{x}\bar{f}(i) = f(\mathbf{x})\cdot\sigma_1\,\bar{f}(\sigma_2)\wedge\bar{f}(\sigma_3) - f(\mathbf{x})\cdot\sigma_2\,\bar{f}(\sigma_1)\wedge\bar{f}(\sigma_3) + f(\mathbf{x})\cdot\sigma_3\,\bar{f}(\sigma_1)\wedge\bar{f}(\sigma_2)\,. \tag{22}$$

Now we combine the wedge products:

$$\mathbf{x}\bar{f}(i) = f(\mathbf{x})\cdot\sigma_1\,\bar{f}(\sigma_2\wedge\sigma_3) - f(\mathbf{x})\cdot\sigma_2\,\bar{f}(\sigma_1\wedge\sigma_3) + f(\mathbf{x})\cdot\sigma_3\,\bar{f}(\sigma_1\wedge\sigma_2)\,. \tag{23}$$

Next, we replace the bivector arguments by their dual equivalents:

$$\mathbf{x}\bar{f}(i) = f(\mathbf{x})\cdot\sigma_1\,\bar{f}(i\sigma_1) + f(\mathbf{x})\cdot\sigma_2\,\bar{f}(i\sigma_2) + f(\mathbf{x})\cdot\sigma_3\,\bar{f}(i\sigma_3)\,. \tag{24}$$

Now we remember that $f(\mathbf{x})\cdot\sigma_k$ is a scalar and that $\bar{f}$ is linear:

$$\begin{aligned}\mathbf{x}\bar{f}(i) &= \bar{f}(f(\mathbf{x})\cdot\sigma_1\,i\sigma_1) + \bar{f}(f(\mathbf{x})\cdot\sigma_2\,i\sigma_2) + \bar{f}(f(\mathbf{x})\cdot\sigma_3\,i\sigma_3)\\ &= \bar{f}(f(\mathbf{x})\cdot\sigma_1\,i\sigma_1 + f(\mathbf{x})\cdot\sigma_2\,i\sigma_2) + f(\mathbf{x})\cdot\sigma_3\,i\sigma_3)\\ &= \bar{f}([f(\mathbf{x})\cdot\sigma_1\sigma_1 + f(\mathbf{x})\cdot\sigma_2\sigma_2 + f(\mathbf{x})\cdot\sigma_3\sigma_3]i)\\ &= \bar{f}(f(\mathbf{x})i)\,.\end{aligned} \tag{25}$$

Hence,

$$\mathbf{x} = \frac{\bar{f}(\mathbf{y}i)}{\bar{f}(i)} = \frac{\bar{f}(\mathbf{y}i)}{i\det f}\,. \tag{26}$$

[1] We proved that a linear function $\bar{f}$ from the range to the domain exists that satisfies the relation (4).

From this we get

$$f^{-1}(\mathbf{y}) = \frac{\bar{f}(\mathbf{y}i)i^{-1}}{\det f}\,. \tag{27}$$

Lemma 1:

Assuming that f is a linear transformation, then

$$\bar{f}(i) = \underline{f}(i)\,. \tag{28}$$

Proof:

Given: f is a linear transformation, then

$$\begin{aligned} i\bar{f}(i) &= \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \cdot \bar{f}(\sigma_1) \wedge \bar{f}(\sigma_2) \wedge \bar{f}(\sigma_3) \\ &= \vdots \\ &= \underline{f}(\sigma_1) \wedge \underline{f}(\sigma_2) \wedge \underline{f}(\sigma_3) \cdot \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &= \underline{f}(\sigma_1 \wedge \sigma_2 \wedge \sigma_3) \cdot \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &= \underline{f}(i) \cdot i \\ &= \underline{f}(i)i\,. \end{aligned} \tag{29}$$

Hence

$$\bar{f}(i) = \underline{f}(i)\,. \tag{30}$$

Page 257–9. (Einstein summation convention in effect where reasonable.)

$$\mathbf{f}_k = f(\sigma_k) = \sigma_j f_{jk} = \mathbf{f}_k \cdot \sigma_j\, \sigma_j\,. \tag{31}$$

Or,

$$\sigma_j f_{jk} = \langle\, \sigma_j f(\sigma_k)\, \rangle = \langle\, \sigma_j \mathbf{f}_k\, \rangle\,. \tag{32}$$

Now,

$$\begin{aligned} \sigma_i \cdot (g f \sigma_k) &= \sigma_i \cdot (g\, f_{jk}\sigma_j) \\ &= f_{jk}\sigma_i \cdot (g\,\sigma_j) \\ &= f_{jk}\sigma_i \cdot (g_{\ell j}\sigma_\ell) \\ &= g_{\ell j} f_{jk}(\sigma_i \cdot \sigma_\ell) \\ &= g_{\ell j} f_{jk}\delta_{i\ell} \\ &= g_{ij} f_{jk}\,. \end{aligned} \tag{33}$$

Problem.

If f is a symmetric linear tranformation, what constraint does that place on $\bar{f}$?
Solution.

Given that f is a symmetric linear tranformation, then its matrix representation is

$$f_{k\ell} = f_{\ell k} \,. \tag{34}$$

This requires that

$$\sigma_k \cdot \mathbf{f}_\ell = \sigma_\ell \cdot \mathbf{f}_k \,, \tag{35}$$

or

$$\sigma_k \cdot f(\sigma_\ell) = \sigma_\ell \cdot f(\sigma_k) \,. \tag{36}$$

On converting the LHS to the adjoint equivalent,

$$\bar{f}(\sigma_k) \cdot \sigma_\ell = \sigma_\ell \cdot f(\sigma_k) \,. \tag{37}$$

Now we multiply through by σ_ℓ and sum

$$\sum \bar{f}(\sigma_k) \cdot \sigma_\ell \sigma_\ell = \sum \sigma_\ell \cdot f(\sigma_k) \sigma_\ell \,. \tag{38}$$

Simplifying, we get

$$\bar{f}(\sigma_k) = f(\sigma_k) \,. \tag{39}$$

♣

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.