Problem 1.1 on Page 259

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1 Problems

On page 259 of NFCM [1], we find Problem (1.1):

Prove that $\underline{f}(\mathbf{x} \wedge \mathbf{y}) = 0$ for $\mathbf{x} \wedge \mathbf{y} \neq 0$ if and only if $f(\mathbf{z}) = 0$ for some nonzero vector $\mathbf{z}$ in the $\mathbf{x} \wedge \mathbf{y}$-plane.

2 Solutions

Part (a): ($\Rightarrow$)

Given $\underline{f}(\mathbf{x} \wedge \mathbf{y}) = 0$ for $\mathbf{x} \wedge \mathbf{y} \neq 0$. Then, if either $f(\mathbf{x}) = 0$ or $f(\mathbf{y}) = 0$, choose $\mathbf{z}$ equal to $\mathbf{x}$ or $\mathbf{y}$ according as the case that gives zero. Otherwise, let

$$\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}\,, \tag{1}$$

where $\mathbf{z}$ is in the $\mathbf{x} \wedge \mathbf{y}$-plane, and where it is not the case that both α and β are zero, so we choose $\alpha \neq 0$. Hence, given that f is a linear function,

$$f(\mathbf{z}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})\,. \tag{2}$$

Then, wedging on the right by $\mathbf{f}(\mathbf{y})$, we get

$$f(\mathbf{z}) \wedge f(\mathbf{y}) = \alpha f(\mathbf{x}) \wedge f(\mathbf{y}) = \alpha \underline{f}(\mathbf{x} \wedge \mathbf{y})\,. \tag{3}$$

Now, since $\underline{f}(\mathbf{x} \wedge \mathbf{y}) = 0$, then

$$f(\mathbf{z}) \wedge f(\mathbf{y}) = 0\,. \tag{4}$$

We have two cases to consider. Either $f(\mathbf{z}) = 0$ (and we're finished) or

$$f(\mathbf{z}) = \gamma f(\mathbf{y})\,. \tag{5}$$

where γ is a nonzero scalar. But if this is true, then

$$f(\mathbf{z}) - \gamma f(\mathbf{y}) = f(\mathbf{z} - \gamma\mathbf{y}) = 0\,. \tag{6}$$

Let $\mathbf{z}' = \mathbf{z} - \gamma\mathbf{y}$. If $\mathbf{z}'$ is in the $\mathbf{x} \wedge \mathbf{y}$-plane then we have $f(\mathbf{z}') = 0$. But $\mathbf{z}'$ is in the $\mathbf{x} \wedge \mathbf{y}$-plane if $\mathbf{z}' \wedge \mathbf{y} \wedge \mathbf{x} = 0$.

$$\mathbf{z}' \wedge \mathbf{x} \wedge \mathbf{y} = (\mathbf{z} - \gamma\mathbf{y}) \wedge \mathbf{x} \wedge \mathbf{y} = \mathbf{z} \wedge \mathbf{y} \wedge \mathbf{x} = 0\,. \tag{7}$$

Therefore, $\mathbf{z}'$ is the vector we are looking for.

Part (b): ($\Leftarrow$) We have a nonzero vector $\mathbf{z}$ in the $\mathbf{x} \wedge \mathbf{y}$-plane, such that $f(\mathbf{z}) = 0$, and, of course, $\mathbf{x} \wedge \mathbf{y} \neq 0$. Show that $\underline{f}(\mathbf{x} \wedge \mathbf{y}) = 0$.

Proof:

We already know that we have a $\mathbf{z}$ such that $f(\mathbf{z}) = 0$ and $\mathbf{z}$ is in the $\mathbf{x} \wedge \mathbf{y}$-plane. Hence, for some α and β, not both zero, and we assume that $\alpha \neq 0$,

$$\mathbf{z} = \alpha \mathbf{x} + \beta \mathbf{y} \,, \tag{8}$$

and where it is not the case that both α and β are zero, so for definiteness, we choose $\alpha \neq 0$. Hence, given that f is a linear function,

$$f(\mathbf{z}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}) \,. \tag{9}$$

Then, wedging on the right by $\mathbf{f}(\mathbf{y})$, we get

$$f(\mathbf{z}) \wedge f(\mathbf{y}) = \alpha f(\mathbf{x}) \wedge f(\mathbf{y}) = \alpha \underline{f}(\mathbf{x} \wedge \mathbf{y}) \,. \tag{10}$$

Now, since $f(\mathbf{z}) = 0$ and $\alpha \neq 0$ then $\underline{f}(\mathbf{x} \wedge \mathbf{y}) = 0$, which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.