Problem 1.8 on Page 260–1

P. Reany

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1 Problems

On page 260–1 of NFCM [1], we find Problem (1.8):

When the elements of an $n \times n$ matrix can be factored in the form $\alpha_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$, the determinant of the matrix can be expressed as

$$\det \alpha_{ij} = \det \mathbf{a}_i \cdot \mathbf{b}_j = (\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n) \,. \tag{1}$$

Show that for a determinant so defined, that:

(a) It changes sign if any two rows are interchanged.

(b) It is unchanged by the interchange of rows and columns.

(c) It vanishes if two rows are equal.

(d) It vanishes if two rows are linearly dependent.

2 Solution

(a) The $\mathbf{a}$ vectors represent the rows. Interchanging two rows is the same as interchanging two vectors in $\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_1$. If the two vectors are next to each other, interchanging them produces a factor of -1. Otherwise, we need to count the transpositions needed to move them both. Let's start with moving the right one to the position of the left one by making only side-by-side transpositions, one after another. Say that requires T transpositions. But that operation leaves the left $\mathbf{a}$ originally on the left now one place closer to where it needs to be moved, and therefore we require only $T-1$ side-by-side transpositions to get to where the original right $\mathbf{a}$ was, for a total of $2T-1$ transpositions to move them both. Therefore, the overall sign change is given by the factor

$$(-1)^{2T-1} = -1 \,. \tag{2}$$

(b) The interchange of rows and columns simultaneously would have the effect of taking the transpose of the matrix, or in the version in Eq. (1), take the reverse of it

$$\begin{aligned}
\det[\alpha_{ji}] &= \langle\, (\mathbf{b}_n \wedge \cdots \wedge \mathbf{b}_1)(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n)\,\rangle \\
&= \langle\, (\mathbf{b}_n \wedge \cdots \wedge \mathbf{b}_1)(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n)\,\rangle^\dagger \\
&= \langle\, (\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_1)(\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n)\,\rangle \\
&= (\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n) \\
&= \det[\alpha_{ij}]
\end{aligned} \tag{3}$$

(c) Show the determinant will vanish if any two rows are equal. This is a corollary to the result of Part (a). If we interchange two equal rows of a (square) matrix, performing an identity operation, the value of the determinant is not changed, yet it must be negated by the result of Part (a). That is only possible if the value of the determinant is zero. The geometric algebra way to prove this is to see that two identical rows if the same as two identical $\mathbf{a}$ vectors in the blade $\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_1$, which will make it zero and thus set the determinant to zero.

(d) We can do better. We can show that the determinant is zero if there is a linear dependence among the rows, which corresponds to a linear dependence of the n $\mathbf{a}$ vectors in the product $\mathbf{a}_n \wedge \cdots \wedge \mathbf{a}_1$. So, let's assume that there is. Therefore, for n scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ (not all zero) the following it true that

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_n \mathbf{a}_n = 0\,. \tag{4}$$

Without loss of generality, let's assume that $\alpha_n \neq 0$, then wedge on the right with $\mathbf{a}_{n-1} \wedge \cdots \wedge \mathbf{a}_1$, so we have that

$$\alpha_n \mathbf{a}_n \wedge \mathbf{a}_{n-1} \wedge \cdots \wedge \mathbf{a}_1 = 0\,, \tag{5}$$

where all the other terms dropped out because there are two identical vectors among the products. But since $\alpha_n \neq 0$ then $\mathbf{a}_n \wedge \mathbf{a}_{n-1} \wedge \cdots \wedge \mathbf{a}_1 = 0$, which makes the determinant vanish.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.