Problem 1.4 on Page 260

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1 Problems

On page 260 of NFCM [1], we find Problem (1.4):
Prove that the following propositions about a linear function f on $\mathcal{E}^3$ are equivalent.

Notational note: $\mho$ = a logical contradiction.

(a) f is nonsinglular.
(b) $f(\mathbf{x}) = 0$ iff $\mathbf{x} = 0$.
(c) To every vector $\mathbf{y}$ there corressponds a unique vector $\mathbf{x}$ such that $\mathbf{y} = f(\mathbf{x})$.

2 Solutions

Part 1): (a) $\Rightarrow$ (b)

Given f is nonsinglular; hence $\underline{f}(i) \neq 0$. Show that

Case 1) $f(\mathbf{x}) = 0 \implies \mathbf{x} = 0$.

Case 2) $\mathbf{x} = 0 \implies f(\mathbf{x}) = 0$.

Case 1) We begin with the assumption that $f(\mathbf{x}) = 0$. Does this allow us to show that $\mathbf{x} = 0$? Yes. Let $\mathbf{x}$ be given as

$$\mathbf{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \,. \tag{1}$$

Then

$$0 = f(\mathbf{x}) = f(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) = x_1 f(\sigma_1) + x_2 f(\sigma_2) + x_3 f(\sigma_3) \,. \tag{2}$$

Wedging this through on the right by $(\sigma_2) \wedge f(\sigma_3)$, we get

$$0 = x_1 f(\sigma_1) \wedge f(\sigma_2) \wedge f(\sigma_3) = x_1 \underline{f}(i) \,. \tag{3}$$

But since $\underline{f}(i) \neq 0$ then $x_1 = 0$. By similar reasoning, we can show that $x_2 = x_3 = 0$, which means that $\mathbf{x} = 0$.

Case 2) Given that $\mathbf{x} = 0$ show that $f(\mathbf{x}) = 0$. Let $\mathbf{y}$ be an arbitrary vector in the range space. Then

$$\mathbf{x} \cdot \bar{f}(\mathbf{y}) = f(\mathbf{x}) \cdot \mathbf{y} = 0 \,. \tag{4}$$

We'll prove this by contradiction. Assume that $f(\mathbf{x}) \neq 0$. Then call $f(\mathbf{x})$ as $\mathbf{y}_1$. Since $f(\mathbf{x}) \cdot \mathbf{y} = 0$ must be true for arbitrary $\mathbf{y}$, we choose $\mathbf{y} = \mathbf{y}_1$. Then,

$$\mathbf{y}_1 \cdot \mathbf{y}_1 = 0 \,. \quad \mho \tag{5}$$

Part 2): (b) $\Rightarrow$ (c)

Given that $\mathbf{x} = 0$ iff $f(\mathbf{x}) = 0$, show that to every $\mathbf{y}$, an arbitrary vector in the range space, there corresponds a unique vector $\mathbf{x}$ in the domain space such that $\mathbf{y} = f(\mathbf{x})$. In other words, show that f is a bijection, being 1-1 and onto.

First we'll show that f is 1-1. To do this, we need to show that whenever $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ then $\mathbf{x}_1 = \mathbf{x}_2$. Therefore, assume that

$$f(\mathbf{x}_1) = f(\mathbf{x}_2)\,. \tag{6}$$

Then

$$f(\mathbf{x}_1) - f(\mathbf{x}_2) = 0\,. \tag{7}$$

But f is linear, so

$$f(\mathbf{x}_1 - \mathbf{x}_2) = 0\,. \tag{8}$$

But by assumption $f(\mathbf{x}_1 - \mathbf{x}_2) = 0$ implies that $\mathbf{x}_1 - \mathbf{x}_2 = 0$. Therefore, $\mathbf{x}_1 = \mathbf{x}_2$ and we have shown that f is 1-1.

Now to show that f is onto. Our f has nullity zero, hence, by the Rank-Nullity Theorem the rank(f) is the same dimension as the base space, which has dimension 3. Therefore, f is onto.

Part 3): (c) $\Rightarrow$ (a)

We are given that to every vector $\mathbf{y}$ there corresponds a unique vector $\mathbf{x}$ such that $\mathbf{y} = f(\mathbf{x})$. Show that therefore f is nonsingular, or that $\underline{f}(i) \neq 0$. We have that

$$\begin{aligned} f(\sigma_1) &= \mathbf{u}_1 \\ f(\sigma_2) &= \mathbf{u}_2 \\ f(\sigma_3) &= \mathbf{u}_3\,. \end{aligned} \tag{9}$$

Therefore,

$$f(\sigma_1) \wedge f(\sigma_2) \wedge f(\sigma_3) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3\,. \tag{10}$$

But, $f(\sigma_1) \wedge f(\sigma_2) \wedge f(\sigma_3) = \underline{f}(\sigma_1 \wedge \sigma_2 \wedge \sigma_3) = \underline{f}(i)$. So,

$$\underline{f}(i) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3\,. \tag{11}$$

Now, $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$ is zero iff these three vectors are linearly dependent. If they are not linearly independent, then , $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3 \neq 0$, and in this case $\underline{f}(i) \neq 0$, and therefore, f is nonsinglular.

So, what does it mean for those vectors to be linearly dependent? It means that there exist three scalars α, β, and γ, not all zero, such that

$$\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3 = 0\,. \tag{12}$$

But if (12) is true, then so is

$$\alpha f(\sigma_1) + \beta f(\sigma_2) + \gamma f(\sigma_3) = 0\,, \tag{13}$$

which can be rewritten as

$$f(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3) = 0\,, \tag{14}$$

But since 0 gets mapped by f to zero and that $\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3$ also gets mapped to zero, and that the domain vector $\mathbf{x}$ is unique for every $\mathbf{y} = \mathbf{f}(\mathbf{x})$, then this implies that

$$\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 = 0\,. \tag{15}$$

But the basis vectors cannot be linearly dependent. $\mho$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.