Problem 1.7 on Page 260

P. Reany

October 9, 2021

1 Problems

On page 260 of NFCM [1], we find Problem (1.7):

Find the inverse the linear transformation

$$g\mathbf{x} = \mathbf{y} = \alpha\mathbf{x} + \mathbf{x}\cdot\mathbf{B} = \alpha\mathbf{x} + \mathbf{b}\times\mathbf{x}\,, \tag{1}$$

where $\mathbf{B} = i\mathbf{b}$ is, of course, a bivector.

2 Solution

In Problem (1.6), we could work with dot products and easily solve for $\mathbf{x}$ as a function of $\mathbf{y}$. But this time, maybe it's better to just get rid of the dot product right away.

Let's begin by restating the problem in the form:

$$\mathbf{y} = \alpha\mathbf{x} + \mathbf{x}\cdot\mathbf{B}\,. \tag{2}$$

But

$$\mathbf{x}\cdot\mathbf{B} = \tfrac{1}{2}(\mathbf{x}\mathbf{B} - \mathbf{B}\mathbf{x})\,. \tag{3}$$

Substituting this into (2), we get

$$\mathbf{y} = \alpha\mathbf{x} + \tfrac{1}{2}(\mathbf{x}\mathbf{B} - \mathbf{B}\mathbf{x})\,. \tag{4}$$

Now we have a choice between solving for $\mathbf{x}$ 'on the left' or 'on the right' — but this is just an intermediate step. Let's begin by solving for $\mathbf{x}$ on the right:

$$(2\alpha - \mathbf{B})\mathbf{x} = 2\mathbf{y} - \mathbf{x}\mathbf{B}\,. \tag{5}$$

Next, we multiply through by $\mathbf{B}$ on the right:

$$(2\alpha - \mathbf{B})\mathbf{x}\mathbf{B} = 2\mathbf{y}\mathbf{B} + |\,\mathbf{B}\,|^2\mathbf{x}\,. \tag{6}$$

But we can solve (5) for $\mathbf{x}\mathbf{B}$ and use it in this last equation to get

$$(2\alpha - \mathbf{B})[2\mathbf{y} - (2\alpha - \mathbf{B})\mathbf{x}] = 2\mathbf{y}\mathbf{B} + |\,\mathbf{B}\,|^2\mathbf{x}\,. \tag{7}$$

Next, we again solve for $\mathbf{x}$ on the right:

$$[(2\alpha - \mathbf{B})^2 + |\,\mathbf{B}\,|^2]\mathbf{x} = 2(2\alpha - \mathbf{B})\mathbf{y} - 2\mathbf{y}\mathbf{B}\,. \tag{8}$$

We're almost there; simplifying, gives us:

$$4\alpha(\alpha - \mathbf{B})\mathbf{x} = 4(\alpha\mathbf{y} - \mathbf{y}\wedge\mathbf{B})\mathbf{y}\,. \tag{9}$$

Finally,

$$g^{-1}\mathbf{y} = \mathbf{x} = (\alpha - \mathbf{B})^{-1}[\mathbf{y} - \alpha^{-1}\mathbf{y}\wedge\mathbf{B}]\,. \tag{10}$$

Okay, my answer is not the book's answer, but, on swapping $\mathbf{x}$ for $\mathbf{y}$, it's close.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.