Problem 1.11 on Page 261–2

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1 Problems

On page 261–2 of NFCM [1], we find Problem (1.11):

What do we do when our set of basis elements are not orthonormal? Say we have such a set: $\{\mathbf{e}_k,\ k = 1, 2, 3\}$. We can still define a pseudoscalar from it by

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = ei\,, \tag{1}$$

where e is the magnitude of this pseudoscalar and i is the usual orthonormal pseudoscalar of $\mathcal{E}^3$. We note that

$$e = i^{-1}\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3\,. \tag{2}$$

We've lost the convenience of taking inner products of orthonormal vectors by the standard

$$\sigma_j \cdot \sigma_k = \delta_{jk}\,, \tag{3}$$

but the next-best thing is to introduce what's called the *reciprocal frame*, defined implicitly by

$$\mathbf{e}^k \cdot \mathbf{e}_j = \delta^k_j\,, \tag{4}$$

if these vectors exist. To prove that they exist, we'll first defined them and then show that they have the required properties.

Given the following vectors

$$\begin{aligned}
\mathbf{e}^1 &= \frac{\mathbf{e}_2 \wedge \mathbf{e}_3}{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3} = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e}\,,\\
\mathbf{e}^2 &= \frac{\mathbf{e}_3 \wedge \mathbf{e}_1}{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3} = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{e}\,,\\
\mathbf{e}^3 &= \frac{\mathbf{e}_1 \wedge \mathbf{e}_2}{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e}\,,
\end{aligned} \tag{5}$$

show that they satisfy (**??**).

2 Solution

Since the proof for each of them is similar to the other two, I'll only work out the first one, which requires us to show that

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1\,, \quad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0\,, \quad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0\,. \tag{6}$$

Alright,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = e^{-1}\mathbf{e}_2 \times \mathbf{e}_3 \cdot \mathbf{e}_1 = e^{-1}e = 1\,. \tag{7}$$

Now, $\mathbf{e}^1 \cdot \mathbf{e}_2 = 0$ and $\mathbf{e}^1 \cdot \mathbf{e}_3 = 0$ iff $\mathbf{e}^1 \cdot \mathbf{e}_2 \wedge \mathbf{e}_3 = 0$

$$\mathbf{e}^1 \cdot \mathbf{e}_2 \wedge \mathbf{e}_3 = \frac{\mathbf{e}_2 \wedge \mathbf{e}_3}{ei} \cdot \mathbf{e}_2 \wedge \mathbf{e}_3 \,. \tag{8}$$

But $\dfrac{\mathbf{e}_2 \wedge \mathbf{e}_3}{ei}$ is a vector along $\mathbf{e}_2 \times \mathbf{e}_3$ and therefore is orthogonal to $\mathbf{e}_2 \wedge \mathbf{e}_3$. Hence,

$$\mathbf{e}^1 \cdot \mathbf{e}_2 \wedge \mathbf{e}_3 = 0 \,. \tag{9}$$

The rest of the problem about finding contravariant components is rather easy.

Uniqueness.

Uniqueness is assured by a simple proof by contradiction. Assume that in addition to the reciprocal frame defined in (5) that satisfy (4), there is a second reciprocal frame $\{\mathbf{e}'^k,\ k = 1, 2, 3\}$ that also satisfies (4). For this second set of reciprocal vectors to be different than the original vectors, at least one pair of corresponding vectors must be different from each other. This being the case, then the vector $\mathbf{e}'^1 - \mathbf{e}^1$ is either exactly zero, or it is not. If it is, then then two vectors are identical. If the vector $\mathbf{e}'^1 - \mathbf{e}^1$ is not zero, then the two vectors that comprise it are distinct, and

$$(\mathbf{e}'^1 - \mathbf{e}^1) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \neq 0 \,. \tag{10}$$

Now, we expand $\mathbf{e}^1 \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$, using the constraints of (4) to get $\mathbf{e}^1 \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_3$. Now, we also expand $\mathbf{e}'^1 \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$, using the constraints of (4) to get $\mathbf{e}'^1 \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_3$. Therefore, we have that

$$(\mathbf{e}'^1 - \mathbf{e}^1) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_3 = 0 \,, \tag{11}$$

which contradicts the assumption that these vectors are distinct. And a similar argument goes for the other two pairs vectors.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.