Problem 1.10 on Page 261

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1 Problems

On page 261 of NFCM [1], we find Problem (1.10):

Let the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be linearly independent over an n-dimensional space, so that

$$A_n \equiv \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n \neq 0\,, \tag{1}$$

and therefore A_n is a pseudoscalar for the space. Given the equation

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_n \mathbf{a}_n = \mathbf{c}\,, \tag{2}$$

where $\mathbf{c} \neq 0$, find a formula for the kth coefficient α_k.

2 Solution

First, let's redo (2) to insert the kth term:

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_k \mathbf{a}_k + \ldots + \alpha_n \mathbf{a}_n = \mathbf{c}\,, \tag{3}$$

Now, wedge through on both sides on the left by $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1}$ and on the right by $\mathbf{a}_{k+1} \wedge \cdots \wedge \mathbf{a}_n$, to get

$$\alpha_k \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1} \wedge (\mathbf{c})_k \wedge \mathbf{a}_{k+1} \wedge \cdots \wedge \mathbf{a}_n\,. \tag{4}$$

And, since $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n$ is a pseudoscalar, we can divide through by it,to get

$$\alpha_k = \frac{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1} \wedge (\mathbf{c})_k \wedge \mathbf{a}_{k+1} \wedge \cdots \wedge \mathbf{a}_n}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n}\,. \tag{5}$$

Now, let $B_n \equiv \mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n$ be any nonzero pseudoscalar. Then, to get the equation

$$\alpha_k = \frac{(\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1} \wedge (\mathbf{c})_k \wedge \mathbf{a}_{k+1} \wedge \cdots \wedge \mathbf{a}_n)}{(\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n) \cdot (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n)}\,, \tag{6}$$

go back to (4) and dot through on the left of both sides by B_n, to get a scalar equation and then divide.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.