Problem 1.12 on Page 262–3

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1 Problems

On page 262–3 of NFCM [1], we find Problem (1.12):

We are now in $\mathcal{E}^n$ space with standard pseudoscalar $i = \sigma_1\sigma_2 \cdots \sigma_n$. The matrix elements of the adjoint operator $\bar{f}$ are given by

$$\bar{f}_{ij} = \sigma_i \cdot (\bar{f}\sigma_j) = \sigma_j \cdot (f\sigma_i) = f_{ji} \,. \tag{1}$$

The transformation of the basis is given by

$$\bar{\mathbf{f}}_k = \bar{f}\sigma_k = \sum_j \sigma_j \bar{f}_{jk} = \sum_j f_{kj}\sigma_j \,. \tag{2}$$

Now, show that the matrix elements of the inverse operator f^{-1} are given by

$$\begin{aligned} f^{-1}_{jk} &= \frac{\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n}{\bar{\mathbf{f}}_1 \wedge \cdots \wedge \bar{\mathbf{f}}_n} \\ &= \frac{(\sigma_1 \wedge \cdots \wedge \sigma_n) \cdot (\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n)}{\det f_{ij}} \,, \end{aligned}$$

where $(\sigma_j)_k$ indicates that $\mathbf{f}_k$ has been replaced by σ_j.

2 Solution

First, we need a formula for the inverse of a linear function. From Eq. (1.22), page 256 of the text, we have

$$f^{-1}(\mathbf{y}) = \bar{f}(\mathbf{y}i)/\bar{f}(i) = \frac{\bar{f}(\mathbf{y}i)i^{-1}}{\det f} \,. \tag{3}$$

Hence,

$$f^{-1}(\sigma_k) = \bar{f}(\sigma_k i)/\bar{f}(i) = \frac{\bar{f}(\sigma_k i)i^{-1}}{\det f} \,. \tag{4}$$

Now,

$$\sigma_k i = (-1)^{k-1}\sigma_1 \wedge \cdots \wedge \check{\sigma}_k \wedge \cdots \wedge \sigma_n \,, \tag{5}$$

so

$$\begin{aligned} \bar{f}(\sigma_k i) &= (-1)^{k-1}\bar{f}(\sigma_1 \wedge \cdots \wedge \check{\sigma}_k \wedge \cdots \wedge \sigma_n) \\ &= (-1)^{k-1}(\bar{\mathbf{f}}_1 \wedge \cdots \wedge \check{\bar{\mathbf{f}}}_k \wedge \cdots \wedge \bar{\mathbf{f}}_n) \end{aligned}$$

Therefore,

$$\begin{aligned}
f_{jk}^{-1} &= \sigma_j \cdot f^{-1}(\sigma_k) \\
&= \sigma_j \cdot \frac{(-1)^{k-1}(\bar{\mathbf{f}}_1 \wedge \cdots \wedge \check{\bar{\mathbf{f}}}_k \wedge \cdots \wedge \bar{\mathbf{f}}_n) i^{-1}}{\det f_{ij}} \\
&= \frac{(-1)^{k-1}\langle\, \sigma_j(\bar{\mathbf{f}}_1 \wedge \cdots \wedge \check{\bar{\mathbf{f}}}_k \wedge \cdots \wedge \bar{\mathbf{f}}_n) i^{-1} \,\rangle}{\det f_{ij}} \\
&= \frac{(\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n) i^{-1}}{\det f_{ij}} \\
&= \frac{(\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n)}{\bar{\mathbf{f}}_1 \wedge \cdots \wedge \bar{\mathbf{f}}_n} \,.
\end{aligned}$$

Okay, now we rewrite this last result as

$$f_{jk}^{-1}(\bar{\mathbf{f}}_1 \wedge \cdots \wedge \bar{\mathbf{f}}_n) = (\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n) \,. \tag{6}$$

Now we dot through on the left by $(\sigma_1 \wedge \cdots \wedge \sigma_n)$, to get

$$f_{jk}^{-1}(\sigma_1 \wedge \cdots \wedge \sigma_n) \cdot (\bar{\mathbf{f}}_1 \wedge \cdots \wedge \bar{\mathbf{f}}_n) = (\sigma_1 \wedge \cdots \wedge \sigma_n) \cdot (\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n) \,. \tag{7}$$

and it immediately follows that

$$f_{jk}^{-1} = \frac{(\sigma_1 \wedge \cdots \wedge \sigma_n) \cdot (\bar{\mathbf{f}}_1 \wedge \cdots \wedge (\sigma_j)_k \wedge \cdots \wedge \bar{\mathbf{f}}_n)}{\det f_{ij}} \,. \tag{8}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.