Notes on Chapter 5, Section 2

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1 Symmetric and Skew-symmetric Operators

These notes cover pages 263 to 275 of NFCM [1].

Maybe this following theorem should be placed in the previous section, but here goes.

Theorem:

If $f : X \to Y$ is 1-1 and onto, then $\bar{f} : Y \to X$ is also 1-1.

Proof:

Given $f : X \to Y$ is 1-1 means that if

$$f(\mathbf{x}_1) = f(\mathbf{x}_2) \quad \text{then} \quad \mathbf{x}_1 = \mathbf{x}_2\,. \tag{1}$$

Now, we need to show that if

$$\bar{f}(\mathbf{y}_1) = \bar{f}(\mathbf{y}_2) \quad \text{then} \quad \mathbf{y}_1 = \mathbf{y}_2\,. \tag{2}$$

We start by picking some nonzero $\mathbf{x}_0 \in X$ and getting from the last equation

$$\mathbf{x}_0 \cdot \bar{f}(\mathbf{y}_1) = \mathbf{x}_0 \cdot \bar{f}(\mathbf{y}_2)\,. \tag{3}$$

But we can flipflop this to get

$$f(\mathbf{x}_0) \cdot \mathbf{y}_1 = f(\mathbf{x}_0) \cdot \mathbf{y}_2\,. \tag{4}$$

From this we get that

$$f(\mathbf{x}_0) \cdot \mathbf{y}_1 - f(\mathbf{x}_0) \cdot \mathbf{y}_2 = 0\,, \tag{5}$$

and therefore

$$f(\mathbf{x}_0) \cdot (\mathbf{y}_1 - \mathbf{y}_2) = 0\,. \tag{6}$$

However, this equation must be true for arbitrary nonzero $\mathbf{x}_0$. The first thing we know is that $f(\mathbf{x}_0)$ is not zero because f is 1-1 and only the zero vector of X gets mapped to the zero vector of Y.

Now, since f is onto, and $\mathbf{x}_0$ is arbitrary, if we assume that $\mathbf{y}_1 - \mathbf{y}_2 \neq 0$, we can always find an $\mathbf{x}_0$ that has a nonzero component of $f(\mathbf{x}_0)$ along $\mathbf{y}_1 - \mathbf{y}_2$, which would contradict the assumption of Eq. (6). Thus, we are forced to conclude that $\mathbf{y}_1 - \mathbf{y}_2 = 0$, and from that, that $\mathbf{y}_1 = \mathbf{y}_2$. Done.

Let $\mathcal{A}$ be a skew-symmetric linear operator. Then we know that $\mathcal{A}$ satisfies the equation

$$\mathcal{A}\mathbf{a}_j = \tfrac{1}{2}(\mathcal{A}\mathbf{a}_j - \overline{\mathcal{A}}\mathbf{a}_j)\,. \tag{7}$$

Dotting this by $\mathbf{a}_k$, we have that

$$\begin{aligned}
\mathbf{a}_k \cdot \mathcal{A}\mathbf{a}_j &= \tfrac{1}{2}(\mathbf{a}_k \cdot \mathcal{A}\mathbf{a}_j - \mathbf{a}_k \cdot \overline{\mathcal{A}}\mathbf{a}_j) \\
&= \tfrac{1}{2}((\overline{\mathcal{A}}\mathbf{a}_k) \cdot \mathbf{a}_j - (\mathcal{A}\mathbf{a}_k) \cdot \mathbf{a}_j) \\
&= -\tfrac{1}{2}((\mathcal{A}\mathbf{a}_k) \cdot \mathbf{a}_j - (\overline{\mathcal{A}}\mathbf{a}_k) \cdot \mathbf{a}_j) \\
&= -\tfrac{1}{2}(\mathbf{a}_j \cdot \mathcal{A}\mathbf{a}_k - \mathbf{a}_j \cdot \overline{\mathcal{A}}\mathbf{a}_k) \\
&= -\mathbf{a}_j \cdot \mathcal{A}\mathbf{a}_k\,.
\end{aligned} \tag{8}$$

If we define the matrix components of $\mathcal{A}$ as $\mathcal{A}_{jk} = \sigma_j \cdot \mathcal{A}\sigma_k$, then it's easy to show that

$$\mathcal{A}_{jk} = -\mathcal{A}_{kj}\,. \tag{9}$$

Now, it should be straightfoward to show that $\mathcal{A}\mathbf{x}$ can be represented as

$$\mathcal{A}\mathbf{x} = \mathbf{x} \cdot \mathbf{A} \tag{10}$$

for some bivector nozero $\mathbf{A}$. If we can make this representaton, we should be able to substitute (10) into the LHS of (8), in the form of $\mathcal{A}\mathbf{a}_j = \mathbf{a}_j \cdot \mathbf{A}$, to get the final RHS:

$$\begin{aligned}
\mathbf{a}_k \cdot \mathcal{A}\mathbf{a}_j &= \mathbf{a}_k \cdot (\mathbf{a}_j \cdot \mathbf{A}) \\
&= (\mathbf{a}_k \wedge \mathbf{a}_j) \cdot \mathbf{A}) \\
&= -(\mathbf{a}_j \wedge \mathbf{a}_k) \cdot \mathbf{A}) \\
&= -\mathbf{a}_j \cdot (\mathbf{a}_k \cdot \mathbf{A}) \\
&= -\mathbf{a}_j \cdot \mathcal{A}\mathbf{a}_k\,.
\end{aligned} \tag{11}$$

Eigenvectors and Eigenvalues

From (2.5), we have that

$$(f - \lambda)\mathbf{e} = 0 \tag{12}$$

shows that the operator $(f - \lambda)$ is singular. This means that

$$\det(f - \lambda) = 0 \tag{13}$$

So,

$$(f-\lambda)(i) = \det(f-\lambda)i = 0\,. \qquad (14)$$

Therefore,

$$(f-\lambda)\sigma_1 \wedge (f-\lambda)\sigma_2 \wedge (f-\lambda)\sigma_3 = 0\,, \qquad (15)$$

or

$$(\mathbf{f}_1-\lambda\sigma_1) \wedge (\mathbf{f}_2-\lambda\sigma_2) \wedge (\mathbf{f}_3-\lambda\sigma_3) = 0\,, \qquad (16)$$

where $\mathbf{f}_k = f\sigma_k$. Hence,

$$\det(f-\lambda) = i^{-1}(\mathbf{f}_1-\lambda\sigma_1) \wedge (\mathbf{f}_2-\lambda\sigma_2) \wedge (\mathbf{f}_3-\lambda\sigma_3) = 0\,, \qquad (17)$$

which is Eq. (2.6).

Now, we go back to Eq. (2.5) [Eq. (12)]. Let's decompose $\mathbf{e}$ into components:

$$\mathbf{e} = \sum_{k=1}^{3} \mathbf{e}\cdot\sigma_k\,\sigma_k = \sum_{k=1}^{3} e_k\sigma_k\,, \qquad (18)$$

where

$$e_k = \mathbf{e}\cdot\sigma_k\,. \qquad (19)$$

Then

$$(f-\lambda)\sum_{k=1}^{3} e_k\sigma_k = 0\,. \qquad (20)$$

Therefore,

$$(f-\lambda)\sum_{k=1}^{3} e_k\sigma_k = \sum_{k=1}^{3}(\mathbf{f}_k-\lambda\sigma_k)e_k = 0\,. \qquad (21)$$

Now, if we define

$$\mathbf{g}_k = \mathbf{f}_k - \lambda\sigma_k\,, \qquad (22)$$

then (21) becomes

$$\mathbf{g}_1 e_1 + \mathbf{g}_2 e_2 + \mathbf{g}_3 e_3 = 0\,. \qquad (23)$$

which is Eq. (2.8a) in the text.

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An operator is said to be 'positive' if

$$\mathbf{x}\cdot(f\mathbf{x}) > 0 \quad \text{for all } \mathbf{x}\,. \qquad (24)$$

Our job now is to prove that all the eigenvalues are positive. We're going to expand $f\mathbf{x}$ and assume that we have an orthonormal basis:

$$\mathbf{x}\cdot(f\mathbf{x}) = \mathbf{x}\cdot\Big[\sum_{k=1}^{3}\lambda_k\,\mathbf{x}\cdot\mathbf{e}_k\,\mathbf{e}_k\Big] > 0\,. \qquad (25)$$

Now, let $\mathbf{x} = \mathbf{e}_j$:

$$\mathbf{e}_j \cdot \big[\sum_{k=1}^{3} \lambda_k \, \mathbf{e}_j \cdot \mathbf{e}_k \, \mathbf{e}_k\big] > 0\,. \tag{26}$$

From this we get that

$$\sum_k \delta_{jk}\lambda_k > 0 \quad \text{for all} \quad j = 1, 2, 3\,, \tag{27}$$

which simplifies to $\lambda_j > 0$ for all $j = 1, 2, 3$.

Let $\mathbf{u}$ be an arbitrary unitvector from the origin in E^3. Let $\mathcal{S}$ be a symmetric operator on E^3. Then

$$\mathcal{S}\mathbf{u} = \mathbf{x} \tag{28}$$

is the set of points that $\mathcal{S}$ maps from the unit sphere to some ellipsoid. We know that $\mathcal{S}$ has the form

$$\mathcal{S}\mathbf{x} = \sum_{k=1}^{3} \lambda_k \, \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k\,. \tag{29}$$

we can derive an equation for the ellipsoid by the following trick.

$$\mathbf{u} = \mathcal{S}^{-1}\mathbf{x} = \sum_{k=1}^{3} \frac{1}{\lambda_k} \, \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k \tag{30}$$

On dotting this last equation with itself, we get

$$1 = (\mathcal{S}^{-1}\mathbf{x}) \cdot (\mathcal{S}^{-1}\mathbf{x})\,. \tag{31}$$

Clearly, the inverse of a symmetric transformation is also a symmetric transformation. Hence,

$$1 = (\mathcal{S}^{-1}\mathbf{x}) \cdot (\mathcal{S}^{-1}\mathbf{x}) = \mathbf{x} \cdot \overline{\mathcal{S}^{-1}}((\mathcal{S}^{-1}\mathbf{x})) = \mathbf{x} \cdot (\mathcal{S}^{-2}\mathbf{x}) \tag{32}$$

or

$$\mathbf{x} \cdot (\mathcal{S}^{-2}\mathbf{x}) = 1\,. \tag{33}$$

Now we expand in a basis, assuming that the basis vectors are orthonormal.

$$\sum_{j=1}^{3} x_j \mathbf{e}_j \cdot \big[\sum_{k=1}^{3} \frac{1}{\lambda_k^2} x_k \mathbf{e}_k\big] = 1\,, \tag{34}$$

or

$$\sum_{j=1}^{3} x_j \big[\sum_{k=1}^{3} \frac{1}{\lambda_k^2} x_k \delta_{jk}\big] = 1\,, \tag{35}$$

from which we get Eq. (2.26) in the text:

$$\frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} + \frac{x_3^2}{\lambda_3^2} = 1\,. \tag{36}$$

Page 272-275: Eigenvalues in 2D: Mohr's Algorithm.

I intend this subsection to be lengthy to fill-in the details and to make-up for the difficulty of my own reading of the printed mathematics, especially the subscripts, which seem to be washed out.

We begin with a positive symmetric operator $\mathcal{S}$ in 2D with eigenvalues and eigenvectors:

$$\mathcal{S}\mathbf{e}_\pm = \lambda_\pm \mathbf{e}_\pm\,, \tag{37}$$

where $\mathbf{e}_\pm$ are principal vectors and $\lambda_\pm$ their respective eigenvalues.

Now, let $\mathbf{u}$ be a unit vector ($\mathbf{u}^2 = 1$) in the plane spanned by the two principal vectors. Let's project $\mathbf{u}$ onto the newly defined unit vector $\mathbf{e} \equiv \hat{\mathbf{e}}_+$

$$\mathbf{u}_\parallel = \mathbf{u}\cdot\mathbf{e}\,\mathbf{e} \quad \text{and thus} \quad \mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\parallel = \mathbf{e}\,\mathbf{e}\wedge\mathbf{u}\,. \tag{38}$$

Therefore,

$$\begin{aligned}\mathcal{S}\mathbf{u} &= \mathcal{S}(\mathbf{u}_\parallel + \mathbf{u}_\perp) = \lambda_+\mathbf{u}_\parallel + \lambda_-\mathbf{u}_\perp \\ &= \lambda_+\mathbf{e}\,\mathbf{e}\cdot\mathbf{u} + \lambda_-\mathbf{e}\,\mathbf{e}\wedge\mathbf{u} \\ &= \tfrac{1}{2}\lambda_+(\mathbf{u} + \mathbf{eue}) + \tfrac{1}{2}\lambda_-(\mathbf{u} - \mathbf{eue})\,,\end{aligned} \tag{39}$$

where we used that

$$\mathbf{e}\cdot\mathbf{u} = \tfrac{1}{2}(\mathbf{eu} + \mathbf{ue})\,, \tag{40a}$$

$$\mathbf{e}\wedge\mathbf{u} = \tfrac{1}{2}(\mathbf{eu} - \mathbf{ue})\,. \tag{40b}$$

From this we get Eq. (2.28) of the text:

$$\mathcal{S}\mathbf{u} = \tfrac{1}{2}(\lambda_+ + \lambda_-)\mathbf{u} + \tfrac{1}{2}(\lambda_+ - \lambda_-)\mathbf{eue}\,. \tag{41}$$

Now, we're going to deal quite a bit with the geometric product of the type $\mathbf{vi} = \mathbf{v}\cdot\mathbf{i}$, where $\mathbf{v}$ is a vector in the $\mathbf{i}$-plane. Hence $\mathbf{v}\wedge\mathbf{i} \equiv 0$ and

$$\mathbf{vi} = -\mathbf{iv}\,. \tag{42}$$

A simple proof of this relies on the fact that $\mathbf{vi}$ is a vector:

$$\mathbf{vi} = (\mathbf{vi})^\dagger = \mathbf{i}^\dagger\mathbf{v}^\dagger = -\mathbf{iv}\,. \tag{43}$$

Therefore, with $\mathbf{e}$ and $\mathbf{u}$ both in the $\mathbf{i}$-plane,

$$\mathbf{ieuie} = -\mathbf{eue}\,. \tag{44}$$

So, now we arrive at the text's Eq. (2.30) by replacing $\mathbf{u}$ by $\mathbf{ui}$ and multiplying through by $\mathbf{i}$ on the left:

$$\begin{aligned}\mathbf{i}\mathcal{S}(\mathbf{ui}) &= \tfrac{1}{2}(\lambda_+ + \lambda_-)\mathbf{iui} + \tfrac{1}{2}(\lambda_+ - \lambda_-)\mathbf{ieuie} \\ &= \tfrac{1}{2}(\lambda_+ + \lambda_-)\mathbf{u} - \tfrac{1}{2}(\lambda_+ - \lambda_-)\mathbf{eue}\,. \end{aligned} \tag{45}$$

From (41) and (45), we can define two new vectors

$$\mathbf{u}_+ = \mathcal{S}\mathbf{u} + \mathbf{i}\mathcal{S}(\mathbf{ui}) = (\lambda_+ + \lambda_-)\mathbf{u}\,, \tag{46a}$$

$$\mathbf{u}_- = \mathcal{S}\mathbf{u} - \mathbf{i}\mathcal{S}(\mathbf{ui}) = (\lambda_+ - \lambda_-)\mathbf{eue}\,, \tag{46b}$$

which are the Eqs. (2.31a) and (2.31b) of the text.

Assuming that $\lambda_+ \geq \lambda_-$, then,

$$|\,\mathbf{u}_\pm\,| = \lambda_+ \pm \lambda_-\,. \tag{47}$$

From this we can solve for the lambdas:

$$\lambda_\pm = \tfrac{1}{2}(|\,\mathbf{u}_+\,| \pm |\,\mathbf{u}_-\,|)\,, \tag{48}$$

which is Eq. (2.33c) in the text.

From (46b), we can define

$$\hat{\mathbf{u}}_- \equiv \mathbf{eue}\,, \tag{49}$$

Then,

$$\mathbf{e}\hat{\mathbf{u}}_- = \mathbf{ue} = e^{\mathbf{i}\phi}\,. \tag{50}$$

which tells us that $\mathbf{e}$ is halfway between the vectors $\hat{\mathbf{u}}_-$ and $\mathbf{u} = \hat{\mathbf{u}}_+$. (Think of $\mathbf{e}$ as along the diagonal of a rhombus whose sides are given by vectors $\hat{\mathbf{u}}_-$ and $\hat{\mathbf{u}}_+$.) Then,

$$\mathbf{e}_+ = \alpha(\hat{\mathbf{u}}_+ + \hat{\mathbf{u}}_-)\,, \tag{51}$$

which is Eq. (2.32) in the text. If $\hat{\mathbf{u}}_+ \wedge \hat{\mathbf{u}}_- \neq 0$, then

$$\mathbf{e}_- = \alpha(\hat{\mathbf{u}}_+ - \hat{\mathbf{u}}_-) \tag{52}$$

(and $\mathbf{e}_-$ is along the other diagonal of that rhombus). That $\mathbf{e}_+ \cdot \mathbf{e}_- = 0$ can be determined by direct computation. These are summarized in Eq. (2.33b).

Thus we can summarize Mohr's Algorithm as the following steps:

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1. Choose a convenient vector $\mathbf{u}$ in the $\mathbf{i}$-plane.
2. Calculate $\mathbf{u}_\pm$ from (46a) and (46b).
3. For $\hat{\mathbf{u}}_+ \wedge \hat{\mathbf{u}}_- \neq 0$, $\mathbf{e}_\pm = \alpha(\hat{\mathbf{u}}_+ \pm \hat{\mathbf{u}}_-)$ are principal vectors of $\mathcal{S}$.
4. Then the principal values are given by $\lambda_\pm = \frac{1}{2}(|\,\mathbf{u}_+\,| \pm |\,\mathbf{u}_-\,|)$, which comes from (46a) and (46b) by taking absolute values.

Now we skip to the example on page 274. We begin with the symmetric operator:[1]

$$\mathcal{S}\mathbf{u} = \mathbf{a}\mathbf{a}\wedge\mathbf{u} + \mathbf{b}\mathbf{b}\wedge\mathbf{u}\,. \tag{53}$$

A quick path to a solution is to consider $\mathcal{S}\mathbf{a}$, given by

$$\mathcal{S}\mathbf{a} = \mathbf{a}\mathbf{a}\wedge\mathbf{a} + \mathbf{b}\mathbf{b}\wedge\mathbf{a} = \mathbf{b}\cdot\mathbf{b}\wedge\mathbf{a} = b^2\mathbf{a} - \mathbf{b}\mathbf{b}\cdot\mathbf{a}\,. \tag{54}$$

And, for $\mathbf{a}\wedge\mathbf{b} = |\,\mathbf{a}\wedge\mathbf{b}\,|\mathbf{i}$,

$$\mathbf{i}\mathcal{S}(\mathbf{a}\mathbf{i}) = \mathbf{i}[\mathbf{a}\mathbf{a}\wedge(\mathbf{a}\mathbf{i}) + \mathbf{b}\mathbf{b}\wedge(\mathbf{a}\mathbf{i})]\,. \tag{55}$$

Now, since $\mathbf{a}\mathbf{i}$ is a vector, then $\mathbf{a}\wedge(\mathbf{a}\mathbf{i})$ is a bivector, hence

$$\mathbf{a}\wedge(\mathbf{a}\mathbf{i}) = \langle\,\mathbf{a}(\mathbf{a}\mathbf{i})\,\rangle_2 = \langle\,(\mathbf{a}\mathbf{a})\mathbf{i}\,\rangle_2 = \mathbf{a}^2\mathbf{i} = a^2\mathbf{i}\,. \tag{56}$$

Similarly,

$$\mathbf{b}\wedge(\mathbf{a}\mathbf{i}) = \langle\,\mathbf{b}(\mathbf{a}\mathbf{i})\,\rangle_2 = \langle\,(\mathbf{b}\mathbf{a})\mathbf{i}\,\rangle_2 = \mathbf{b}\cdot\mathbf{a}\,\mathbf{i}\,. \tag{57}$$

Then, on substituting these results into (55), we have

$$\mathbf{i}\mathcal{S}(\mathbf{a}\mathbf{i}) = \mathbf{i}[\mathbf{a}\mathbf{a}^2\mathbf{i} + \mathbf{b}\mathbf{b}\cdot\mathbf{a}\,\mathbf{i}] = \mathbf{a}^3 + \mathbf{b}\mathbf{b}\cdot\mathbf{a}\,. \tag{58}$$

On setting $\mathbf{u}$ in (46a) and in (46b) to $\hat{\mathbf{a}}$, and using the results of (54) and (58), we get

$$\mathbf{a}_+ = [b^2\hat{\mathbf{a}} - \mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}] + [\mathbf{a}^2\hat{\mathbf{a}} + \mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}] = (a^2+b^2)\hat{\mathbf{a}}\,, \tag{59a}$$

$$\mathbf{a}_- = [b^2\hat{\mathbf{a}} - \mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}] - [\mathbf{a}^2\hat{\mathbf{a}} + \mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}] = (b^2-a^2)\hat{\mathbf{a}} - 2\mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}\,, \tag{59b}$$

which are the results at the bottom of page 274.

From this we easily get that

$$|\,\mathbf{a}_+\,| = a^2 + b^2\,. \tag{60}$$

But deriving $|\,\mathbf{a}_-\,|$ is a little trickier.

$$\begin{aligned}|\,\mathbf{a}_-\,|^2 &= [(b^2-a^2)\hat{\mathbf{a}} - 2\mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}]\cdot[(b^2-a^2)\hat{\mathbf{a}} - 2\mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}]\\ &= (b^2-a^2)^2 - 4(b^2-a^2)(\mathbf{b}\cdot\hat{\mathbf{a}})^2 + 4b^2(\mathbf{b}\cdot\hat{\mathbf{a}})^2\\ &= (b^2-a^2)^2 + 4(\mathbf{b}\cdot\mathbf{a})^2\,.\end{aligned} \tag{61a}$$

Obviously, then

$$|\,\mathbf{a}_-\,| = [\,(b^2-a^2)^2 + 4(\mathbf{a}\cdot\mathbf{b})^2\,]^{1/2}\,, \tag{62}$$

which gives us the equation at the top of page 275. Using (48), we get

$$\lambda_\pm = \tfrac{1}{2}\left(a^2 + b^2 \pm [\,(b^2-a^2)^2 + 4(\mathbf{a}\cdot\mathbf{b})^2\,]^{1/2}\right)\,, \tag{63}$$

which is Eq. (2.39) of the text, though I get with it an overall factor of 1/2.

[1] This can be proved symmetric by showng that $\mathbf{v}\cdot\mathcal{S}\mathbf{u} = \mathbf{u}\cdot\mathcal{S}\mathbf{v}$.

Now to calculate the principal vectors. From (51) and (52), we get, respectively,

$$\mathbf{e}_+ = \alpha(\hat{\mathbf{a}}_+ + \hat{\mathbf{a}}_-)\,, \tag{64a}$$
$$\mathbf{e}_- = \alpha(\hat{\mathbf{a}}_+ - \hat{\mathbf{a}}_-)\,. \tag{64b}$$

Thus,

$$\mathbf{e}_\pm = \alpha(\hat{\mathbf{a}}_+ \pm \hat{\mathbf{a}}_-) = \alpha\Big[\hat{\mathbf{a}} \pm \frac{(b^2 - a^2)\hat{\mathbf{a}} - 2\mathbf{b}\mathbf{b}\cdot\hat{\mathbf{a}}}{[\,(b^2-a^2)^2 + 4(\mathbf{a}\cdot\mathbf{b})^2\,]^{1/2}}\Big]\,. \tag{65}$$

However, we are free to scale this as we please, so we'll set $\alpha = a$, to get

$$\mathbf{e}_\pm = \mathbf{a} \pm \frac{(b^2 - a^2)\mathbf{a} - 2\mathbf{b}\mathbf{b}\cdot\mathbf{a}}{[\,(b^2-a^2)^2 + 4(\mathbf{a}\cdot\mathbf{b})^2\,]^{1/2}}\,. \tag{66}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.