Select Problems for Chapter 5, Section 2, Pages 275–277

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Note: It may be helpful to study my notes for the entire chapter before studying the exercise solutions here.

1 Problem (2.1)

Find the adjoint as well as the symmetric and skew-symmetric parts of the linear transformation

$$f\mathbf{x} = \alpha\mathbf{x} + \mathbf{a}\,\mathbf{b}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{A}\,. \tag{1}$$

On dotting this last equation by $\mathbf{y}$, we get

$$\mathbf{y}\cdot f\mathbf{x} = \alpha\mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} + \mathbf{y}\cdot(\mathbf{x}\cdot\mathbf{A})\,. \tag{2}$$

Now, we flip-flop between f and its adjoint $\bar{f}$:[1]

$$\begin{aligned}
\bar{f}\,(\mathbf{y})\cdot\mathbf{x} &= \alpha\mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} + \mathbf{y}\wedge\mathbf{x}\cdot\mathbf{A} \\
&= \alpha\mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} - \mathbf{x}\wedge\mathbf{y}\cdot\mathbf{A} \\
&= \alpha\mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} - \mathbf{x}\cdot(\mathbf{y}\cdot\mathbf{A}) \\
&= \alpha\mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} - (\mathbf{y}\cdot\mathbf{A})\cdot\mathbf{x} \\
&= [\,\alpha\mathbf{y} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b} - (\mathbf{y}\cdot\mathbf{A})\,]\cdot\mathbf{x} \\
&= [\,\alpha\mathbf{y} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b} + (\mathbf{A}\cdot\mathbf{y})\,]\cdot\mathbf{x}\,.
\end{aligned} \tag{3}$$

Now, since this last equation must be true for arbitrary $\mathbf{x}$ and $\mathbf{y}$, we can conclude that

$$\bar{f}\,(\mathbf{y}) = \alpha\mathbf{y} + \mathbf{y}\cdot\mathbf{a}\,\mathbf{b} + \mathbf{A}\cdot\mathbf{y}\,, \tag{4}$$

or, in terms of $\mathbf{x}$,

$$\bar{f}\,(\mathbf{x}) = \alpha\mathbf{x} + \mathbf{x}\cdot\mathbf{a}\,\mathbf{b} + \mathbf{A}\cdot\mathbf{x}\,. \tag{5}$$

Now that we have both f and $\bar{f}$, we can determine f_+ and f_-.

$$\begin{aligned}
f_+(\mathbf{x}) &= \tfrac{1}{2}(f(\mathbf{x}) + \bar{f}\,(\mathbf{x})) \\
&= \tfrac{1}{2}\big([\alpha\mathbf{x} + \mathbf{a}\,\mathbf{b}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{A}] + [\alpha\mathbf{x} + \mathbf{x}\cdot\mathbf{a}\,\mathbf{b} + \mathbf{A}\cdot\mathbf{x}]\big) \\
&= \alpha\mathbf{x} + \tfrac{1}{2}[\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{x}\,\mathbf{b}]\,.
\end{aligned} \tag{6}$$

[1] We are using that $f\mathbf{x} = \underline{f}\mathbf{x}$.

And then,

$$\begin{aligned} f_{-}(\mathbf{x}) &= \tfrac{1}{2}(f(\mathbf{x}) - \bar{f}(\mathbf{x})) \\ &= \tfrac{1}{2}\big([\alpha\mathbf{x} + \mathbf{a}\,\mathbf{b}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{A}] - [\alpha\mathbf{x} + \mathbf{x}\cdot\mathbf{a}\,\mathbf{b} + \mathbf{A}\cdot\mathbf{x}]\big) \\ &= \tfrac{1}{2}[\mathbf{a}\,\mathbf{b}\cdot\mathbf{x} - \mathbf{a}\cdot\mathbf{x}\,\mathbf{b} + 2\mathbf{x}\cdot\mathbf{A}]\,. \end{aligned} \tag{7}$$

2 Problem (2.4)

We write $\mathcal{S}^n$ for the n-fold product of $\mathcal{S}$ with itself. Prove that if $\mathcal{S}$ is symmetric, with eigenvalues λ_k, then $\mathcal{S}^n$ is also symmetric with eigenvalues λ_k^n and having the same eigenvectors as does $\mathcal{S}$. ♣

We begin with the equation

$$\mathcal{S}\mathbf{e}_k = \lambda_k\mathbf{e}_k \quad (\text{no sum on } k)\,, \tag{8}$$

and we are to show that

$$\mathcal{S}^n\mathbf{e}_k = \lambda_k^n\mathbf{e}_k \quad (\text{no sum on } k)\,, \tag{9}$$

Proof by induction. So,

$$\mathcal{S}^2\mathbf{e}_k = \mathcal{S}(\mathcal{S}\mathbf{e}_k) = \mathcal{S}(\lambda_k\mathbf{e}_k) = \lambda_k\mathcal{S}\mathbf{e}_k = \lambda_k^2\mathbf{e}_k \quad (\text{no sum on } k)\,. \tag{10}$$

And that concludes the base case. Now, we assume that for all $m \le n$ that

$$\mathcal{S}^m\mathbf{e}_k = \lambda_k^m\mathbf{e}_k \quad (\text{no sum on } k)\,, \tag{11}$$

and then show that the relation holds for the power $m+1$ as well. Then,

$$\mathcal{S}(\mathcal{S}^m\mathbf{e}_k) = \mathcal{S}(\lambda_k^m\mathbf{e}_k) \quad (\text{no sum on } k)\,, \tag{12}$$

from which we get

$$\mathcal{S}^{m+1}\mathbf{e}_k = \lambda_k^{m+1}\mathbf{e}_k \quad (\text{no sum on } k)\,, \tag{13}$$

which is what we needed to show.

3 Problem (2.5)

An linear operator $\mathcal{S}$ is given by

$$\begin{aligned} \mathcal{S}\sigma_1 &= 7\sigma_1 + 2\sigma_2 \;+\; 0\,, \\ \mathcal{S}\sigma_2 &= 2\sigma_1 + 6\sigma_2 - 2\sigma_3\,, \\ \mathcal{S}\sigma_3 &= \;0\; - 2\sigma_2 \;+ 5\sigma_3\,. \end{aligned} \tag{14}$$

Find the eigenvalues and eigenvectors. ♣

We find the eigenvalues and eigenvectors by solving for the unknowns in the equation

$$\mathcal{S}(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3) = \lambda(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)\,. \tag{15}$$

But

$$\begin{aligned}\mathcal{S}(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3) &= \alpha\mathcal{S}(\sigma_1) + \beta\mathcal{S}(\sigma_2) + \gamma\mathcal{S}(\sigma_3) \\ &= \alpha(7\sigma_1 + 2\sigma_2) + \beta(2\sigma_1 + 6\sigma_2 - 2\sigma_3) \\ &\qquad + \gamma(-2\sigma_2 \ + 5\sigma_3)\,.\end{aligned} \tag{16}$$

On combining these last two equations, we get

$$\begin{aligned}\lambda\alpha\sigma_1 + \lambda\beta\sigma_2 + \lambda\gamma\sigma_3 &= \alpha(7\sigma_1 + 2\sigma_2) + \beta(2\sigma_1 + 6\sigma_2 - 2\sigma_3) \\ &\qquad + \gamma(-2\sigma_2 \ + 5\sigma_3)\,.\end{aligned} \tag{17}$$

This last equation has to be resolved component-wise:

$$\sigma_1 : \lambda\alpha = 7\alpha + 2\beta + \ 0\,, \tag{18a}$$
$$\sigma_2 : \lambda\beta = 2\alpha + 6\beta - 2\gamma\,, \tag{18b}$$
$$\sigma_3 : \lambda\gamma = \ 0 \ - 2\beta + 5\gamma\,. \tag{18c}$$

And this can be put into matrix form:

$$\begin{pmatrix} 7-\lambda & 2 & 0 \\ 2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,. \tag{19}$$

Given that the last matrix equation is homogeneous, the 3×3 matrix has zero determinant:

$$\begin{vmatrix} 7-\lambda & 2 & 0 \\ 2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0\,, \tag{20}$$

which gives us the characteristic polynomial

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0\,. \tag{21}$$

The roots to this equation are the eigenvalues we're looking for. Wolfram Alpha gives the three roots to be

$$\lambda_1 = 3\,, \qquad \lambda_2 = 6\,, \qquad \lambda_3 = 9\,. \tag{22}$$

We'll proceed to find an eigenvector that corresponds to the first stated eigenvalue. But since the length of this vector is not uniquely determined, we are free to set one of its components as we please (with the possible exception of zero).

So, setting $\lambda = 3$ in (23), we have

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{23}$$

Choosing $\gamma = 1$, then in the last row, we must have that $\beta = 1$. And setting $\beta = 1$ in the first row requires us to set $\alpha = -1/2$. We are free to rescale these values to get for the eigenvector for the first eigenvalue to be $-1, 2, 2$, or

$$\mathbf{e}_1 = -\sigma_1 + 2\sigma_2 + 2\sigma_3 . \tag{24}$$

Note that these values are consistent with the equation derived from the second row. Anyway, the second and third eigenvectors follow similarly.

4 Problem (2.9)

Let $\mathcal{S}$ be a symmetric operator on $\mathcal{R}^3$ to itself, where we assume the eigenvectors $\mathbf{e}_k$ are normalized. Then

$$\mathcal{S}\mathbf{x} = \sum_{k=1}^{3} \lambda_k \, \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k . \tag{25}$$

Show that there exists an inverse operator $\mathcal{S}^{-1}$ to $\mathcal{S}$ which satisfies the following equation

$$\mathcal{S}^{-1}\mathbf{x} = \sum_{k=1}^{3} \frac{1}{\lambda_k} \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k . \tag{26}$$

♣

Our plan is to assume the existence of a symmetric operator $\mathcal{T}$ on $\mathcal{R}^3$ to itself that satisfies the following properties

$$\mathbf{x} = \mathcal{T}\mathcal{S}\mathbf{x} \quad \text{and} \quad \mathbf{x} = \mathcal{S}\mathcal{T}\mathbf{x} \quad \text{for all } \mathbf{x} . \tag{27}$$

Clearly, such an operator acts like an inverse for $\mathcal{S}$. Starting with

$$\mathbf{x} = \mathcal{T}\mathcal{S}\mathbf{x} \tag{28}$$

and then expanding $\mathbf{x}$ by

$$\mathbf{x} = \sum_{k=1}^{3} \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k . \tag{29}$$

we get

$$\begin{aligned} \sum_{k=1}^{3} \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k &= \mathcal{T}\mathcal{S}\mathbf{x} = \mathcal{T}\,[\,\sum_{k=1}^{3} \lambda_k \, \mathbf{x} \cdot \mathbf{e}_k \, \mathbf{e}_k\,] \\ &= \sum_{k=1}^{3} \lambda_k \, \mathbf{x} \cdot \mathbf{e}_k \mathcal{T}\,[\,\mathbf{e}_k\,] . \end{aligned} \tag{30}$$

This last equation can be expressed in the alternative form

$$\sum_{k=1}^{3} \mathbf{x} \cdot \mathbf{e}_k \left[\, \mathbf{e}_k - \lambda_k \mathcal{T}\left(\mathbf{e}_k\right) \right] = 0\,. \tag{31}$$

For this to be true for arbitrary $\mathbf{x}$, we must have that for each k:

$$\mathbf{e}_k - \lambda_k \mathcal{T}\left(\mathbf{e}_k\right) = 0\,, \tag{32}$$

whose solution for $\mathcal{T}(\mathbf{e}_k)$ is

$$\mathcal{T}\mathbf{e}_k = \frac{1}{\lambda_k}\, \mathbf{e}_k\,. \tag{33}$$

Hence,

$$\mathcal{T}\mathbf{x} = \sum_{k=1}^{3} \frac{1}{\lambda_k}\, \mathbf{x} \cdot \mathbf{e}_k\, \mathbf{e}_k\,. \tag{34}$$

Now that we have the form for $\mathcal{T}\mathbf{x}$, it's a simple matter to demonstrate that

$$\mathbf{x} = \mathcal{S}\mathcal{T}\mathbf{x} \quad \text{for all } \mathbf{x}\,. \tag{35}$$

And with that accomplished, the proof is finished.

5 Problem (2.10)

Find the eigenvalues and eigenvectors of the tensors

$$\text{A)}\ \mathcal{S}\mathbf{u} = \mathbf{a}\,\mathbf{a} \cdot \mathbf{u} + \mathbf{b}\,\mathbf{b} \cdot \mathbf{u}\,, \tag{36a}$$

$$\text{B)}\ \mathcal{T}\mathbf{u} = \mathbf{a}\,\mathbf{b} \cdot \mathbf{u} + \mathbf{b}\,\mathbf{a} \cdot \mathbf{u}\,. \tag{36b}$$

Part A)

So, we use Mohr's algorithm, beginning with

$$\mathcal{S}\mathbf{u} = \mathbf{a}\,\mathbf{a} \cdot \mathbf{u} + \mathbf{b}\,\mathbf{b} \cdot \mathbf{u}\,, \tag{37}$$

into which we replace $\mathbf{u}$ by $\mathbf{a}$, to get

$$\mathcal{S}\hat{\mathbf{a}} = a^2 \hat{\mathbf{a}} + \hat{\mathbf{a}} \cdot \mathbf{b}\,\mathbf{b}\,. \tag{38}$$

To this we add the relation

$$\begin{aligned}
\mathbf{i}\mathcal{S}(\hat{\mathbf{a}}\mathbf{i}) &= \mathbf{i}[\,\mathbf{a}\,\mathbf{a} \cdot (\hat{\mathbf{a}}\mathbf{i}) + \mathbf{b}\,\mathbf{b} \cdot (\hat{\mathbf{a}}\mathbf{i})\,] \\
&= \mathbf{i}[\,\mathbf{b}\,\mathbf{b} \wedge \mathbf{a}\mathbf{i}\,] \\
&= \mathbf{b}\,\mathbf{b} \wedge \hat{\mathbf{a}} \\
&= \mathbf{b} \cdot \mathbf{b} \wedge \hat{\mathbf{a}} = b^2 \hat{\mathbf{a}} - \hat{\mathbf{a}} \cdot \mathbf{b}\,\mathbf{b}\,,
\end{aligned} \tag{39}$$

where we used that the vectors $\mathbf{a}$ and $\mathbf{ai}$ are orthogonal to each other, and that

$$\mathbf{b}\cdot(\hat{\mathbf{a}}\mathbf{i}) = \langle\,\mathbf{b}\cdot(\hat{\mathbf{a}}\mathbf{i})\,\rangle = \langle\,(\mathbf{b}\wedge\hat{\mathbf{a}})\cdot\mathbf{i}\,\rangle = \langle\,(\mathbf{b}\wedge\hat{\mathbf{a}})\mathbf{i}\,\rangle = \mathbf{b}\wedge\hat{\mathbf{a}}\,\mathbf{i}\,. \tag{40}$$

The paired vectors $\mathbf{a}_+$ and $\mathbf{a}_-$ are given by

$$\begin{aligned}\mathbf{a}_+ &= \mathcal{S}\hat{\mathbf{a}} + \mathbf{i}\mathcal{S}(\hat{\mathbf{a}}\mathbf{i}) = (a^2\hat{\mathbf{a}} + \hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b}) + (b^2\hat{\mathbf{a}} - \hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b})\\ &= (a^2+b^2)\hat{\mathbf{a}}\,, \qquad (41a)\\ \mathbf{a}_- &= \mathcal{S}\hat{\mathbf{a}} - \mathbf{i}\mathcal{S}(\hat{\mathbf{a}}\mathbf{i}) = (a^2\hat{\mathbf{a}} + \hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b}) - (b^2\hat{\mathbf{a}} - \hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b})\\ &= (a^2-b^2)\hat{\mathbf{a}} + 2\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b}\,. \qquad (41b)\end{aligned}$$

Thus, we get

$$\begin{aligned}|\,\mathbf{a}_+\,| &= a^2+b^2 \qquad (42a)\\ |\,\mathbf{a}_-\,|^2 &= [\,(a^2-b^2)\hat{\mathbf{a}} + 2\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b}\,]\cdot[\,(a^2-b^2)\hat{\mathbf{a}} + 2\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b}\,] \qquad (42b)\\ |\,\mathbf{a}_-\,| &= [\,(a^2-b^2)^2 + 4(a^2-b^2)(\hat{\mathbf{a}}\cdot\mathbf{b})^2 + 4b^2(\hat{\mathbf{a}}\cdot\mathbf{b})^2\,]^{1/2} \qquad (42c)\\ &= [\,(a^2-b^2)^2 + 4a^2(\hat{\mathbf{a}}\cdot\mathbf{b})^2\,]^{1/2}\,. \qquad (42d)\end{aligned}$$

Now, we can calculate the eigenvalue by

$$\lambda_\pm = \tfrac{1}{2}\big(\,|\,\mathbf{a}_+\,| \pm |\,\mathbf{a}_-\,|\,\big)\,. \tag{43}$$

From this we get that

$$\lambda_\pm = \tfrac{1}{2}\Big[\,(a^2+b^2) \pm [\,(a^2-b^2)^2 + 4a^2(\hat{\mathbf{a}}\cdot\mathbf{b})^2\,]^{1/2}\,\Big]\,. \tag{44}$$

So, eigenvectors $\mathbf{e}_\pm$ are given by

$$\mathbf{e}_\pm = \alpha\big(\hat{\mathbf{a}}_+ + \hat{\mathbf{a}}_-\big) = \alpha\Big[\,\hat{\mathbf{a}} \pm \frac{(a^2-b^2)\hat{\mathbf{a}} + 2\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{b}}{[\,(a^2-b^2)^2 + 4a^2(\hat{\mathbf{a}}\cdot\mathbf{b})^2\,]^{1/2}}\,\Big]\,. \tag{45}$$

Setting $\alpha = a$, we get

$$\mathbf{e}_\pm = \mathbf{a} \pm \frac{(a^2-b^2)\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b}\,\mathbf{b}}{[\,(a^2-b^2)^2 + 4a^2(\hat{\mathbf{a}}\cdot\mathbf{b})^2\,]^{1/2}}\,. \tag{46}$$

Part B)

So, we again use Mohr's algorithm, beginning with

$$\mathcal{T}\mathbf{u} = \mathbf{a}\,\mathbf{b}\cdot\mathbf{u} + \mathbf{b}\,\mathbf{a}\cdot\mathbf{u}\,, \tag{47}$$

into which we replace $\mathbf{u}$ by $\hat{\mathbf{a}}$, to get

$$\mathcal{T}\hat{\mathbf{a}} = \mathbf{ab}\cdot\hat{\mathbf{a}} + \mathbf{b}\,\mathbf{a}\cdot\hat{\mathbf{a}} = \hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{a} + a\mathbf{b}\,. \tag{48}$$

To this we add the relation

$$
\begin{aligned}
\mathbf{i}\mathcal{T}(\hat{\mathbf{a}}\mathbf{i}) &= \mathbf{i}[\,\mathbf{a}\,\mathbf{b}\cdot(\hat{\mathbf{a}}\mathbf{i})+\mathbf{b}\,\mathbf{a}\cdot(\hat{\mathbf{a}}\mathbf{i})\,] \\
&= \mathbf{i}[\,\mathbf{a}\cdot\mathbf{b}\wedge\hat{\mathbf{a}}\mathbf{i}+0\,] \\
&= \mathbf{a}\cdot\mathbf{b}\,\hat{\mathbf{a}} - a\mathbf{b}\,,
\end{aligned} \tag{49}
$$

where we used that $\mathbf{b}\cdot(\hat{\mathbf{a}}\mathbf{i}) = \langle\,\mathbf{b}\hat{\mathbf{a}}\mathbf{i}\,\rangle = \mathbf{b}\wedge\hat{\mathbf{a}}\mathbf{i}$.

The paired vectors $\mathbf{a}_+$ and $\mathbf{a}_-$ are given by

$$
\mathbf{a}_+ = \mathcal{T}\hat{\mathbf{a}} + \mathbf{i}\mathcal{T}(\hat{\mathbf{a}}\mathbf{i}) = (\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{a}+a\mathbf{b}) + (\mathbf{a}\,\mathbf{b}\cdot\hat{\mathbf{a}} - a\mathbf{b}) = 2\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{a}\,, \tag{50a}
$$

$$
\mathbf{a}_- = \mathcal{T}\hat{\mathbf{a}} - \mathbf{i}\mathcal{T}(\hat{\mathbf{a}}\mathbf{i}) = (\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{a}+a\mathbf{b}) - (\mathbf{a}\,\mathbf{b}\cdot\hat{\mathbf{a}} - a\mathbf{b}) = 2a\mathbf{b}\,. \tag{50b}
$$

Hence, we get

$$
|\,\mathbf{a}_+\,| = 2\,|\,\mathbf{a}\cdot\mathbf{b}\,|\;, \tag{51a}
$$

$$
|\,\mathbf{a}_-\,| = 2ab\,. \tag{51b}
$$

Now, we can calculate the eigenvalues by

$$
\lambda_\pm = \tfrac{1}{2}\big(\,|\,\mathbf{a}_+\,| \pm |\,\mathbf{a}_-\,|\,\big)\,. \tag{52}
$$

From this we get that

$$
\lambda_\pm = \tfrac{1}{2}[\,2\,|\,\mathbf{a}\cdot\mathbf{b}\,| \pm 2ab\,] = |\,\mathbf{a}\cdot\mathbf{b}\,| \pm ab\,. \tag{53}
$$

And the eigenvectors $\mathbf{e}_+$ are given by

$$
\mathbf{e}_\pm = \alpha\big(\hat{\mathbf{a}}_+ \pm \hat{\mathbf{a}}_-\big) = \alpha\Big[\,\frac{2\hat{\mathbf{a}}\cdot\mathbf{b}\,\mathbf{a}}{2\,|\,\mathbf{a}\cdot\mathbf{b}\,|} \pm \hat{\mathbf{b}}\Big]\,. \tag{54}
$$

Setting $\alpha = a$, we get

$$
\mathbf{e}_\pm = \operatorname{sgn}(\mathbf{b}\cdot\mathbf{a})\,\hat{\mathbf{a}} \pm a\hat{\mathbf{b}}\,. \tag{55}
$$

6 Problem (2.11)

For an operator f specified by the symmetric matrix

$$
[\,f\,] = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \tag{56}
$$

with respect to an orthonormal basis σ_1 and σ_2, show that

$$
\lambda_\pm = \tfrac{1}{2}[\,f_{11}+f_{22}\,] \pm \tfrac{1}{2}\,\big|\,(f_{11}-f_{22})^2 + 4f_{12}^2\,\big|^{1/2} \tag{57}
$$

are eigenvalues, and the angle ϕ to the vector $\mathbf{e}_+$ is

$$
\tan 2\phi = -\frac{2f_{12}}{f_{22}-f_{11}}\,. \tag{58}
$$

♣

We begin with

$$f(\sigma_1) = f_{11}\sigma_1 + f_{12}\sigma_2\,, \tag{59a}$$
$$f(\sigma_2) = f_{21}\sigma_1 + f_{22}\sigma_2\,. \tag{59b}$$

But we are told that f is symmetric, hence, $f_{21} = f_{12}$. Therefore, the last equations become

$$f(\sigma_1) = f_{11}\sigma_1 + f_{12}\sigma_2\,, \tag{60a}$$
$$f(\sigma_2) = f_{12}\sigma_1 + f_{22}\sigma_2\,. \tag{60b}$$

We may as well use σ_1 as our special vector. Then,

$$\sigma_+ = f(\sigma_1) + \mathbf{i}f(\sigma_1\mathbf{i})\,. \tag{61}$$

But $\sigma_1\mathbf{i} = \sigma_2$, therefore,

$$\sigma_+ = f(\sigma_1) + \mathbf{i}f(\sigma_2)\,. \tag{62}$$

Now, since

$$\mathbf{i}f(\sigma_2) = \mathbf{i}(f_{12}\sigma_1 + f_{22}\sigma_2) = f_{22}\sigma_1 - f_{12}\sigma_2\,. \tag{63}$$

Therefore,

$$\sigma_+ = (f_{11}\sigma_1 + f_{12}\sigma_2) + (f_{22}\sigma_1 - f_{12}\sigma_2) = (f_{11} + f_{12})\sigma_1\,, \tag{64}$$

and, simnilarly,

$$\sigma_- = (f_{11}\sigma_1 + f_{12}\sigma_2) - (f_{22}\sigma_1 - f_{12}\sigma_2) = (f_{11} - f_{12})\sigma_1 + 2f_{12}\sigma_2\,. \tag{65}$$

Next,

$$|\,\sigma_+\,| = |\,f_{11} + f_{12}\,| \tag{66a}$$
$$|\,\sigma_-\,| = [\,(f_{11} - f_{22})^2 + 4f_{12}^2\,]^{1/2} \tag{66b}$$

Therefore,

$$\lambda_\pm = \tfrac{1}{2}[\,f_{11} + f_{22}\,] \pm \tfrac{1}{2}\left|\,(f_{11} - f_{22})^2 + 4f_{12}^2\,\right|^{1/2}\,. \tag{67}$$

Now, to the angle ϕ. The equation we need form the text is given on page 273, Eq. (2.35):

$$\mathbf{i}\tan 2\phi = \frac{\mathbf{u}_+ \wedge \mathbf{u}_-}{\mathbf{u}_+ \cdot \mathbf{u}_-}\,. \tag{68}$$

But in our case this becomes

$$\mathbf{i}\tan 2\phi = \frac{\sigma_+ \wedge \sigma_-}{\sigma_+ \cdot \sigma_-}\,. \tag{69}$$

Thus,

$$\frac{\sigma_+ \wedge \sigma_-}{\sigma_+ \cdot \sigma_-} = \frac{(f_{11}+f_{12})\sigma_1 \wedge [\,(f_{11}-f_{22})\sigma_1 + 2f_{12}\sigma_2\,]}{(f_{11}+f_{12})\sigma_1 \cdot [\,(f_{11}-f_{22})\sigma_1 + 2f_{12}\sigma_2\,]}\,, \tag{70}$$

which reduces to

$$\frac{\sigma_+ \wedge \sigma_-}{\sigma_+ \cdot \sigma_-} = -\frac{2f_{12}}{f_{22}-f_{11}}\mathbf{i}\,. \tag{71}$$

On combining these last three equations, we get that

$$\tan 2\phi = -\frac{2f_{12}}{f_{22}-f_{11}}\,. \tag{72}$$

7 Problem (2.12)

Solve the eigenvector problem for the tensor

$$\mathcal{S}\mathbf{u} = \mathbf{a}\,\mathbf{a}\wedge\mathbf{u} + \mathbf{b}\,\mathbf{b}\wedge\mathbf{u} + \mathbf{c}\,\mathbf{c}\wedge\mathbf{u}\,, \tag{73}$$

where $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. ♣

Since $\mathbf{c} = -\mathbf{a}-\mathbf{b}$ then

$$\begin{aligned}
\mathcal{S}\hat{\mathbf{a}} &= \mathbf{a}\,\mathbf{a}\wedge\hat{\mathbf{a}} + \mathbf{b}\,\mathbf{b}\wedge\hat{\mathbf{a}} + (-\mathbf{a}-\mathbf{b})\,(-\mathbf{a}-\mathbf{b})\wedge\hat{\mathbf{a}} \\
&= \mathbf{b}\cdot\mathbf{b}\wedge\hat{\mathbf{a}} + (\mathbf{a}+\mathbf{b})\,(\mathbf{a}+\mathbf{b})\wedge\hat{\mathbf{a}} \\
&= \mathbf{b}\cdot\mathbf{b}\wedge\hat{\mathbf{a}} + (\mathbf{a}+\mathbf{b})\cdot\mathbf{b}\wedge\hat{\mathbf{a}} \\
&= 2\mathbf{b}\cdot\mathbf{b}\wedge\hat{\mathbf{a}} + \mathbf{a}\cdot\mathbf{b}\wedge\hat{\mathbf{a}} \\
&= (2b^2 + \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{a}} - (a + 2\hat{\mathbf{a}}\cdot\mathbf{b})\mathbf{b}\,,
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
\mathbf{i}\mathcal{S}(\hat{\mathbf{a}}\mathbf{i}) &= \mathbf{i}\,[\,\mathbf{a}\,\mathbf{a}\wedge(\hat{\mathbf{a}}\mathbf{i}) + \mathbf{b}\,\mathbf{b}\wedge(\hat{\mathbf{a}}\mathbf{i}) + (\mathbf{a}+\mathbf{b})\,(\mathbf{a}+\mathbf{b})\wedge(\hat{\mathbf{a}}\mathbf{i})\,] \\
&= \mathbf{i}\,[\,a^2\hat{\mathbf{a}}\,\mathbf{i} + \mathbf{b}\cdot\hat{\mathbf{a}}\,\mathbf{i} + (\mathbf{a}+\mathbf{b})\,\{\mathbf{a}\wedge(\hat{\mathbf{a}}\mathbf{i}) + \mathbf{b}\wedge(\hat{\mathbf{a}}\mathbf{i})\}\,] \\
&= \mathbf{i}\,[\,a^2\hat{\mathbf{a}}\,\mathbf{i} + 2\mathbf{b}\cdot\hat{\mathbf{a}}\,\mathbf{i} + a(\mathbf{a}+\mathbf{b})\,\mathbf{i} + \mathbf{a}\,\mathbf{b}\wedge(\hat{\mathbf{a}}\mathbf{i})\}\,] \\
&= \mathbf{i}\,[\,a^2\hat{\mathbf{a}}\,\mathbf{i} + 2\mathbf{b}\cdot\hat{\mathbf{a}}\,\mathbf{i} + a(\mathbf{a}+\mathbf{b})\,\mathbf{i} + \mathbf{a}\cdot\mathbf{b}(\hat{\mathbf{a}}\mathbf{i})\,] \\
&= (2a^2 + \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{a}} + (a + 2\hat{\mathbf{a}}\cdot\mathbf{b})\mathbf{b}\,.
\end{aligned} \tag{75}$$

Now,

$$\mathbf{a}_\pm = \mathcal{S}\hat{\mathbf{a}} \pm \mathbf{i}\mathcal{S}(\hat{\mathbf{a}}\mathbf{i})\,. \tag{76}$$

On substituting in, we have that

$$\mathbf{a}_\pm = [\,(2b^2 + \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{a}} - (a + 2\hat{\mathbf{a}}\cdot\mathbf{b})\mathbf{b}\,] \pm [\,(2a^2 + \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{a}} + (a + 2\hat{\mathbf{a}}\cdot\mathbf{b})\mathbf{b}\,]\,. \tag{77}$$

Therefore,

$$\mathbf{a}_+ = C_1\,\hat{\mathbf{a}}\,, \tag{78}$$

where

$$C_1 \equiv 2a^2 + 2b^2 + 2\mathbf{a} \cdot \mathbf{b}\,. \tag{79}$$

And

$$\mathbf{a}_- = C_2\,\hat{\mathbf{a}} - C_3\,\mathbf{b}\,, \tag{80}$$

where

$$C_2 \equiv -2a^2 + 2b^2 \quad \text{and} \quad C_3 \equiv 2a + 4\hat{\mathbf{a}} \cdot \mathbf{b}\,. \tag{81}$$

Therefore

$$|\,\mathbf{a}_+\,| = |\,C_1\,|\;, \tag{82}$$

and

$$|\,\mathbf{a}_-\,| = [\,C_2^2 + 2C_2C_3\,\hat{\mathbf{a}} \cdot \mathbf{b} + C_3^2 b^2\,]^{1/2}\,. \tag{83}$$

Now, we can calculate the eigenvalues by

$$\lambda_\pm = \tfrac{1}{2}\big(\,|\,\mathbf{a}_+\,| \pm |\,\mathbf{a}_-\,|\,\big)\,. \tag{84}$$

Finally,

$$\mathbf{e}_\pm = \alpha\big(\hat{\mathbf{a}}_+ + \hat{\mathbf{a}}_-\big)\,. \tag{85}$$

I leave it to the reader to do the substitutions if he or she wants to.