Problems from Chapter 5, Section 3

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1 Reflections and Rotations

Problem (3.1) page 293 of NFCM [1].

Show that the transformation $\mathcal{R}\mathbf{x} = \mathbf{u}^{-1}\mathbf{x}\mathbf{u}$, determined by a nonzero vector $\mathbf{u}$, is a rotation. Find the axis, the angle and the spinor for this rotation.

For simplicity, let $\mathbf{u}^2 = 1$. Given that $\mathcal{R}\mathbf{x} = \mathbf{u}^{-1}\mathbf{x}\mathbf{u} = \mathbf{u}\mathbf{x}\mathbf{u}$, we know that

$$\mathcal{R}\mathbf{x} = \mathbf{u}\mathbf{x}\mathbf{u} = \mathbf{u}(\mathbf{x}_\| + \mathbf{x}_\perp)\mathbf{u} = \mathbf{x}_\| - \mathbf{x}_\perp\,, \tag{1}$$

where $\mathbf{x}_\|$ commutes with $\mathbf{u}$. So, we want to construct a spinor R that has the same effect on $\mathbf{x}$ as $\mathcal{R}$ does. This spinor will rotate in the plane orthogonal to $\mathbf{u}$. A simple guess would lead us to $R = e^{\frac{1}{2}i\mathbf{u}\pi}$. Let's check it.

$$\begin{aligned} R^\dagger \mathbf{x} R &= e^{-\frac{1}{2}i\mathbf{u}\pi}(\mathbf{x}_\| + \mathbf{x}_\perp)e^{\frac{1}{2}i\mathbf{u}\pi} \\ &= \mathbf{x}_\| + \mathbf{x}_\perp e^{i\mathbf{u}\pi} \\ &= \mathbf{x}_\| - \mathbf{x}_\perp\,. \end{aligned} \tag{2}$$

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Problem (3.2) page 293 of NFCM. Find the inverse of a reflection.

Let the reflection be given by

$$\mathcal{U}\mathbf{x} = -\mathbf{u}\mathbf{x}\mathbf{u} \quad \text{with} \quad \mathbf{u}^2 = 1\,, \tag{3}$$

which reflects $\mathbf{x}$ through the plane orthogonal to $\mathbf{u}$. It's easy to show that the inverse of $\mathcal{U}$ is $\mathcal{U}$:

$$\mathcal{U}\mathcal{U}\mathbf{x} = \mathcal{U}(-\mathbf{u}\mathbf{x}\mathbf{u}) = -\mathbf{u}(-\mathbf{u}\mathbf{x}\mathbf{u})\mathbf{u} = \mathbf{x} \tag{4}$$

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Problem (3.3) page 293 of NFCM. Prove that the product of three successive elementary reflections in orthogonal planes is an inversion, the linear transformation that reverses the direction of every incoming vector.

If you hit a racketball just right into the corner of a racketball court, it can come back at you. Let the normal to the first wall it hits be $\boldsymbol{\sigma}_1$, the next wall be $\boldsymbol{\sigma}_2$, and the last be $\boldsymbol{\sigma}_3$. Each wall the ball hits will reflect it according to the rule

$$\mathbf{x} \to -\boldsymbol{\sigma}_i \mathbf{x} \boldsymbol{\sigma}_i \,. \tag{5}$$

After the three such successive hits (reflections), we get

$$\begin{aligned} \mathbf{x} &\to (-1)^3 \boldsymbol{\sigma}_3 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1 \mathbf{x} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \\ &= -(\pm) i^\dagger \mathbf{x} (\pm) i \,, \end{aligned} \tag{6}$$

where i is the usual right-handed pseudoscalar and $i^\dagger = -i$. Now, if $i = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3$, then we have set-up our system of axes as righthanded and we use the plus sign in (6); otherwise, we must use the minus sign. In either case, we get, remembering that the pseudoscalar commutes with vectors,

$$\mathbf{x} \to -\mathbf{x} \,. \tag{7}$$

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Problem (3.4) page 293 of NFCM. A unitary spinor R can be given the following parameterizations

$$R = e^{(1/2) i \mathbf{a}} = \alpha + i \boldsymbol{\beta} = \alpha (1 + i \boldsymbol{\gamma}) = \frac{1 + i \mathbf{b}}{1 - i \mathbf{b}} \,, \tag{8}$$

where $\mathbf{a}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\mathbf{b}$ are all vectors. Establish the following relations among the parameters:

$$\alpha = \cos \tfrac{1}{2} \mathbf{a} = \frac{1}{\sqrt{1 + \gamma^2}} = \frac{1 - \mathbf{b}^2}{1 + \mathbf{b}^2} \,, \tag{9a}$$

$$\boldsymbol{\gamma} = \tan \tfrac{1}{2} \mathbf{a} = \hat{\mathbf{a}} \tan \tfrac{1}{2} a = \frac{2 \mathbf{b}}{1 - \mathbf{b}^2} \,, \tag{9b}$$

$$\mathbf{b} = \tan \tfrac{1}{4} \mathbf{a} \,. \tag{9c}$$

We already know that

$$e^{(1/2) i \mathbf{a}} = \cos \tfrac{1}{2} \mathbf{a} + i \sin \tfrac{1}{2} \mathbf{a} \,, \tag{10}$$

and this gives us the first identity $\alpha = \cos \frac{1}{2} \mathbf{a}$. We can add the relation for $\boldsymbol{\beta}$

$$\boldsymbol{\beta} = \sin \tfrac{1}{2} \mathbf{a} \,. \tag{11}$$

To establish the relation between α and $\gamma = |\,\boldsymbol{\gamma}\,|$, we can use the fact that R is unitary, that is, $R^\dagger R = 1$. Hence

$$(\alpha (1 + i \boldsymbol{\gamma}))^\dagger (\alpha (1 + i \boldsymbol{\gamma}) = 1 \,, \tag{12}$$

from which we get that

$$\alpha^2 (1 + \gamma^2) = 1 \,. \tag{13}$$

If we stipulate that $\alpha > 0$, then we get

$$\alpha = \frac{1}{\sqrt{1+\gamma^2}} \,. \tag{14}$$

Now we get relations between α and $\boldsymbol{\beta}$ and $\mathbf{b}$. Since $1+\mathbf{b}^2$ is never zero

$$\alpha + i\boldsymbol{\beta} = \frac{1+i\mathbf{b}}{1-i\mathbf{b}} = \frac{1+i\mathbf{b}}{1-i\mathbf{b}}\frac{1+i\mathbf{b}}{1+i\mathbf{b}} = \frac{(1+i\mathbf{b})^2}{1+\mathbf{b}^2} = \frac{1-\mathbf{b}^2+2i\mathbf{b}}{1+\mathbf{b}^2} \,, \tag{15}$$

thus

$$\alpha = \frac{1-\mathbf{b}^2}{1+\mathbf{b}^2} \,, \tag{16a}$$

$$\boldsymbol{\beta} = \frac{2\mathbf{b}}{1+\mathbf{b}^2} \,. \tag{16b}$$

We can get $\boldsymbol{\gamma}$ as a function of $\mathbf{b}$ because $\boldsymbol{\gamma} = \boldsymbol{\beta}/\alpha$ (from Eq. (8)):

$$\boldsymbol{\gamma} = \frac{2\mathbf{b}}{1+\mathbf{b}^2}\bigg/\frac{1-\mathbf{b}^2}{1+\mathbf{b}^2} = \frac{2\mathbf{b}}{1-\mathbf{b}^2} \,. \tag{17}$$

We can also use the trigonometric forms for $\mathbf{b}$ and α to get the trigonometric form for $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma} = \frac{\boldsymbol{\beta}}{\alpha} = \frac{\sin\frac{1}{2}\mathbf{a}}{\cos\frac{1}{2}\mathbf{a}} = \tan\tfrac{1}{2}\mathbf{a} = \hat{\mathbf{a}}\tan\tfrac{1}{2}a \,. \tag{18}$$

We end with establishing the trickiest identity, namely (9c). On eliminating $\boldsymbol{\gamma}$ between the last two equations, we have that

$$\frac{2\mathbf{b}}{1-\mathbf{b}^2} = \tan\tfrac{1}{2}\mathbf{a} \,, \tag{19}$$

or

$$\mathbf{b} = \tfrac{1}{2}(1-\mathbf{b}^2)\tan\tfrac{1}{2}\mathbf{a} \,. \tag{20}$$

Now we solve (16a) for $\mathbf{b}^2$ in terms of α (and we use that $\alpha = \cos\frac{1}{2}\mathbf{a}$):

$$\mathbf{b}^2 = \frac{1-\cos\frac{1}{2}\mathbf{a}}{1+\cos\frac{1}{2}\mathbf{a}} \,. \tag{21}$$

Hence,

$$1-\mathbf{b}^2 = \frac{2\cos\frac{1}{2}\mathbf{a}}{1+\cos\frac{1}{2}\mathbf{a}} \,. \tag{22}$$

Substituting this into (20), we get

$$\mathbf{b} = \frac{\cos\frac{1}{2}\mathbf{a}}{1+\cos\frac{1}{2}\mathbf{a}}\tan\tfrac{1}{2}\mathbf{a} = \frac{\sin\frac{1}{2}\mathbf{a}}{1+\cos\frac{1}{2}\mathbf{a}} = \tan\tfrac{1}{4}\mathbf{a} \,, \tag{23}$$

which I completed by use of a trigonometric identity.
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Problem (3.5) page 293 of NFCM. Given

$$\mathbf{x}' = \mathcal{R}\mathbf{x} = R^\dagger \mathbf{x}\, R\,, \tag{24}$$

derive the "Rodrigues formula"

$$\mathbf{x}' - \mathbf{x} = \boldsymbol{\gamma} \times (\mathbf{x}' + \mathbf{x})\,. \tag{25}$$

My plan is to prove the equivalent form:

$$i(\mathbf{x}' - \mathbf{x}) = i\boldsymbol{\gamma} \times (\mathbf{x}' + \mathbf{x}) = \boldsymbol{\gamma} \wedge (\mathbf{x}' + \mathbf{x}) = \langle\, \boldsymbol{\gamma}(\mathbf{x}' + \mathbf{x})\,\rangle_2\,. \tag{26}$$

I'll do this by expanding the LHS and then the RHS and hope they meet in the middle. We note that $\alpha^2(1+\gamma^2) = 1$, and that $\langle\, \boldsymbol{\gamma}\mathbf{x}_\perp\,\rangle_2 = \boldsymbol{\gamma} \wedge \mathbf{x}$.

$$\begin{aligned}
i(\mathbf{x}' - \mathbf{x}) &= i(R^\dagger \mathbf{x} R - \mathbf{x}) \\
&= i[\,\alpha^2(1 - i\boldsymbol{\gamma})(\mathbf{x}_\parallel + \mathbf{x}_\perp)(1 + i\boldsymbol{\gamma}) - \mathbf{x}\,] \\
&= i[\,\alpha^2(1 - i\boldsymbol{\gamma})(1 + i\boldsymbol{\gamma})\mathbf{x}_\parallel + \alpha^2(1 - i\boldsymbol{\gamma})^2\mathbf{x}_\perp - \mathbf{x}\,] \\
&= i[\,\alpha^2(1 + \gamma^2)\mathbf{x}_\parallel + \alpha^2(1 - \gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp - \mathbf{x}\,] \\
&= i[\,\mathbf{x}_\parallel + \alpha^2(1 - \gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp - \mathbf{x}\,] \\
&= i[\,\alpha^2(1 - \gamma^2 - 2i\boldsymbol{\gamma}) - 1]\mathbf{x}_\perp \\
&= i[\,\alpha^2(1 + \gamma^2) - \alpha^2(2\gamma^2 + 2i\boldsymbol{\gamma}) - 1]\mathbf{x}_\perp \\
&= i[\,-\alpha^2(2\gamma^2 + 2i\boldsymbol{\gamma})]\mathbf{x}_\perp \\
&= -2\alpha^2\gamma^2 i\mathbf{x}_\perp + 2\alpha^2\boldsymbol{\gamma} \wedge \mathbf{x}\,.
\end{aligned} \tag{27}$$

Now we expand the RHS:

$$\begin{aligned}
\langle\, \boldsymbol{\gamma}(\mathbf{x}' + \mathbf{x})\,\rangle_2 &= \langle\, \boldsymbol{\gamma}(R^\dagger \mathbf{x} R + \mathbf{x})\,\rangle_2 \\
&= \langle\, \boldsymbol{\gamma}(\alpha^2(1 - i\boldsymbol{\gamma})(\mathbf{x}_\parallel + \mathbf{x}_\perp)(1 + i\boldsymbol{\gamma}) + \mathbf{x})\,\rangle_2 \\
&= \langle\, \boldsymbol{\gamma}[\,\alpha^2(1 + \gamma^2)\mathbf{x}_\parallel + \alpha^2(1 - \gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp + \mathbf{x})\,]\,\rangle_2 \\
&= \langle\, \boldsymbol{\gamma}[\,\mathbf{x}_\parallel + \alpha^2(1 - \gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp + \mathbf{x})\,]\,\rangle_2 \\
&= \cancel{\langle\, \boldsymbol{\gamma}\mathbf{x}_\parallel\,\rangle_2} + \alpha^2\langle\, \boldsymbol{\gamma}(1 - \gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp\,\rangle_2 + \langle\, \boldsymbol{\gamma}\mathbf{x}\,\rangle_2 \\
&= \alpha^2\langle\, \boldsymbol{\gamma}(1 - \gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp\,\rangle_2 + \boldsymbol{\gamma} \wedge \mathbf{x} \\
&= \alpha^2\langle\, \boldsymbol{\gamma}[\,(1 + \gamma^2) + \boldsymbol{\gamma}(-2\gamma^2 - 2i\boldsymbol{\gamma})\,]\mathbf{x}_\perp\,\rangle_2 + \boldsymbol{\gamma} \wedge \mathbf{x} \\
&= \langle\, \boldsymbol{\gamma}\mathbf{x}_\perp\,\rangle_2 + \alpha^2\langle\, \boldsymbol{\gamma}(-2\gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp\,\rangle_2 + \boldsymbol{\gamma} \wedge \mathbf{x} \\
&= \alpha^2\langle\, \boldsymbol{\gamma}(-2\gamma^2 - 2i\boldsymbol{\gamma})\mathbf{x}_\perp\,\rangle_2 + 2\boldsymbol{\gamma} \wedge \mathbf{x} \\
&= -2\gamma^2\alpha^2\langle\, \boldsymbol{\gamma}\mathbf{x}_\perp\,\rangle_2 - 2\alpha^2\gamma^2\langle\, i\mathbf{x}_\perp\,\rangle_2 + 2\boldsymbol{\gamma} \wedge \mathbf{x} \\
&= -2\gamma^2\alpha^2\boldsymbol{\gamma} \wedge \mathbf{x} - 2\alpha^2\gamma^2 i\mathbf{x}_\perp + 2\boldsymbol{\gamma} \wedge \mathbf{x} \\
&= 2(1 - \gamma^2\alpha^2)\boldsymbol{\gamma} \wedge \mathbf{x} - 2\alpha^2\gamma^2 i\mathbf{x}_\perp\,. \\
&= 2\alpha^2\boldsymbol{\gamma} \wedge \mathbf{x} - 2\alpha^2\gamma^2 i\mathbf{x}_\perp\,.
\end{aligned} \tag{28}$$

Thus, we have achieved the results we needed, though I'm sure there's a much simpler proof than the one I've presented here. I was also able to prove an alternative proof by establishing that

$$\langle\, (i - \boldsymbol{\gamma})\mathbf{x}'\,\rangle_2 = i\mathbf{x} + \langle\, \boldsymbol{\gamma}\mathbf{x}\,\rangle_2\,, \tag{29}$$

but this was no shorter a proof.
♣

Problem (3.6) page 293 of NFCM. Establish the following vector forms for a rotation:

$$\mathbf{x}' = \mathbf{x} + 2\alpha\boldsymbol{\beta}\times\mathbf{x} + 2\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\mathbf{x}) \tag{30a}$$

$$= \mathbf{x} + \hat{\mathbf{a}}\times\mathbf{x}\,\sin a + \hat{\mathbf{a}}\times(\hat{\mathbf{a}}\times\mathbf{x})(1-\cos a)\,. \tag{30b}$$

We'll solve for (30a) first. As is standard in these types of problems, we'll decompose $\mathbf{x}$ into

$$\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}\,, \tag{31}$$

where, in this case,

$$\begin{aligned}\mathbf{x}_{\parallel}\boldsymbol{\beta} &= \boldsymbol{\beta}\mathbf{x}_{\parallel}\,,\\ \mathbf{x}_{\perp}\boldsymbol{\beta} &= -\boldsymbol{\beta}\mathbf{x}_{\perp}\,.\end{aligned}$$

We'll also need that

$$\mathbf{a}\times\mathbf{b} = -i\mathbf{a}\wedge\mathbf{b}\,. \tag{32}$$

This time we'll set $R = \alpha + i\boldsymbol{\beta}$, from which we get that

$$\begin{aligned}&\alpha^2 + \beta^2 = 1\,,\\ &\alpha^2 - \beta^2 = 1 - 2\boldsymbol{\beta}^2\,,\\ &\boldsymbol{\beta}\,\boldsymbol{\beta}\cdot\mathbf{x} = \beta^2\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}\cdot\mathbf{x} = \beta^2\,\mathbf{x}_{\parallel}\,.\end{aligned}$$

We'll also need

$$\begin{aligned}\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\mathbf{x}) &= \boldsymbol{\beta}\,\boldsymbol{\beta}\cdot\mathbf{x} - \mathbf{x}\,\beta^2\\ &= \beta^2\,\mathbf{x}_{\parallel} - \beta^2\mathbf{x}\\ &= -\beta^2\,\mathbf{x}_{\perp}\,.\end{aligned} \tag{33}$$

Now we can to establish (30a):

$$\begin{aligned}\mathbf{x}' &= (\alpha - i\boldsymbol{\beta})\mathbf{x}(\alpha - i\boldsymbol{\beta})\\ &= (\alpha - i\boldsymbol{\beta})(\mathbf{x}_{\parallel} + \mathbf{x}_{\perp})(\alpha - i\boldsymbol{\beta})\\ &= (\alpha - i\boldsymbol{\beta})(\alpha + i\boldsymbol{\beta})\mathbf{x}_{\parallel} + (\alpha - i\boldsymbol{\beta})(\alpha - i\boldsymbol{\beta})\mathbf{x}_{\perp}\\ &= (\alpha^2 + \beta^2)\mathbf{x}_{\parallel} + [(\alpha^2 - \beta^2) - 2i\alpha\boldsymbol{\beta}\,]\mathbf{x}_{\perp}\\ &= \mathbf{x}_{\parallel} + [(1 - 2\boldsymbol{\beta}^2) - 2i\alpha\boldsymbol{\beta}\,]\mathbf{x}_{\perp}\\ &= \mathbf{x} - 2\beta^2\mathbf{x}_{\perp} - 2i\alpha\boldsymbol{\beta}\,\mathbf{x}_{\perp}\\ &= \mathbf{x} - 2\beta^2\mathbf{x}_{\perp} + 2\alpha\boldsymbol{\beta}\times\mathbf{x}_{\perp}\\ &= \mathbf{x} + 2\alpha\boldsymbol{\beta}\times\mathbf{x} + 2\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\mathbf{x})\,.\end{aligned} \tag{34}$$

We'll now solve for (30b). This time we have that

$$\mathbf{x}_{\parallel}\mathbf{a} = \mathbf{a}\mathbf{x}_{\parallel} ,$$
$$\mathbf{x}_{\perp}\mathbf{a} = -\mathbf{a}\mathbf{x}_{\perp} ,$$

and that

$$\hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{x}) = -\mathbf{x}_{\perp} . \tag{35}$$

Now we're ready to establish (30a). But given that the angles in the answer are **not** half angles, it behooves us to formulate the spinors to remove the half-angles as soon as possible.

With

$$R = e^{\frac{1}{2} i \mathbf{a}} , \tag{36}$$

we begin with

$$\begin{aligned}
\mathbf{x}' &= R^{\dagger}\mathbf{x}\, R \\
&= R^{\dagger}(\mathbf{x}_{\parallel} + \mathbf{x}_{\perp})R \\
&= R^{\dagger}R\,\mathbf{x}_{\parallel} + (R^{\dagger})^2\mathbf{x}_{\perp} \\
&= \mathbf{x}_{\parallel} + (e^{-i\mathbf{a}})\mathbf{x}_{\perp} \\
&= \mathbf{x}_{\parallel} + (\mathbf{x}_{\perp} - \mathbf{x}_{\perp}) + (e^{-i\mathbf{a}})\mathbf{x}_{\perp} \\
&= \mathbf{x} + (e^{-i\mathbf{a}} - 1)\mathbf{x}_{\perp} \\
&= \mathbf{x} + (\cos a - i \sin \mathbf{a} - 1)\mathbf{x}_{\perp} \\
&= \mathbf{x} - (1 - \cos a)\mathbf{x}_{\perp} - i\hat{\mathbf{a}}\,\mathbf{x}_{\perp} \sin\, a \\
&= \mathbf{x} + \hat{\mathbf{a}} \times \mathbf{x}\, \sin a + \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{x})(1 - \cos a) ,
\end{aligned}$$

where I reversed the order of the last two terms.
♣

Problem (3.7) page 293 of NFCM. Derive the following expression for the matrix elements of a rotation by an arbitrary vector angle $\mathbf{a}$:

$$e_{jk} = \delta_{jk} \cos a - \epsilon_{jkm}\hat{a}_m \sin a + \hat{a}_j \hat{a}_k (1 - \cos a) , \tag{37}$$

where $\epsilon_{jkm} = i^{\dagger}\boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k \wedge \boldsymbol{\sigma}_m$ and $\hat{a}_k = \hat{\mathbf{a}} \cdot \boldsymbol{\sigma}_k$, which are the direction cosines of the rotation axis.

Basically, we can think of this problem as a corollary to the last problem.

$$\mathbf{x}' = \mathbf{x} + \hat{\mathbf{a}} \times \mathbf{x}\, \sin a + \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{x})(1 - \cos a) . \tag{38}$$

First, we replace $\mathbf{x}$ by $\boldsymbol{\sigma}_k$ to get

$$\mathbf{e}_k = \boldsymbol{\sigma}_k + \hat{\mathbf{a}} \times \boldsymbol{\sigma}_k\, \sin a + \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \boldsymbol{\sigma}_k)(1 - \cos a) . \tag{39}$$

Then, since $e_{jk} = \langle\, \boldsymbol{\sigma}_j \mathbf{e}_k \,\rangle$,

$$e_{jk} = \langle\, \boldsymbol{\sigma}_j [\, \boldsymbol{\sigma}_k + \hat{\mathbf{a}} \times \boldsymbol{\sigma}_k \, \sin a + \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \boldsymbol{\sigma}_k)(1 - \cos a)] \,\rangle \,. \tag{40}$$

Let's look at the parts, starting with the second term:

$$\begin{aligned}
\langle\, \boldsymbol{\sigma}_j [\, \hat{\mathbf{a}} \times \boldsymbol{\sigma}_k \, \sin a] \,\rangle &= \boldsymbol{\sigma}_j \cdot \hat{\mathbf{a}} \times \boldsymbol{\sigma}_k \, \sin a \\
&= \hat{a}_m \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_m \times \boldsymbol{\sigma}_k \, \sin a \\
&= -\hat{a}_m \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k \times \boldsymbol{\sigma}_m \, \sin a \\
&= -\epsilon_{jkm} \hat{a}_m \, \sin a \,.
\end{aligned}$$

It may seem like I pulled a fast one here, but I didn't. It's true that

$$\hat{\mathbf{a}} = \sum_m \hat{a}_m \boldsymbol{\sigma}_m \,, \tag{41}$$

but since $\boldsymbol{\sigma}_j \cdot \hat{\mathbf{a}} \times \boldsymbol{\sigma}_k$ is antisymmetric in the indices, only the particular value of the subscript m that is different from both j and k will survive. I called it m, though m was a dummy variable.

The third term goes as follows:

$$\begin{aligned}
\langle\, \boldsymbol{\sigma}_j [\, \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \boldsymbol{\sigma}_k)(1 - \cos a)] \,\rangle &= \boldsymbol{\sigma}_j \cdot [\hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \boldsymbol{\sigma}_k)(1 - \cos a)] \\
&= \boldsymbol{\sigma}_j \cdot [(\hat{\mathbf{a}}\, \hat{\mathbf{a}} \cdot \boldsymbol{\sigma}_k - \boldsymbol{\sigma}_k)(1 - \cos a)] \\
&= (\hat{a}_j \, \hat{a}_k - \delta_{jk})(1 - \cos a) \,.
\end{aligned}$$

Substituting all this back into (40), we get

$$e_{jk} = \delta_{jk} \cos a - \epsilon_{jkm} \hat{a}_m \sin a + \hat{a}_j \hat{a}_k (1 - \cos a) \,. \tag{42}$$

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Problem (3.9) page 294 of NFCM. Show that any unimodular spinor R can be written in the form

$$R = \pm(\mathbf{u}\mathbf{v})^{1/2} = \frac{1 + \mathbf{u}\mathbf{v}}{[2(1 + \mathbf{u} \cdot \mathbf{v})]^{1/2}} \,, \tag{43}$$

where $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Then derive the trigonometric relations

$$\cos \tfrac{1}{2} a = \frac{1 + \cos a}{[2(1 + \cos a)]^{1/2}} \,, \tag{44a}$$

$$\sin \tfrac{1}{2} a = \frac{\sin a}{[2(1 + \cos a)]^{1/2}} \,. \tag{44b}$$

We'll aproach the solution from the assumption that

$$\pm(\mathbf{u}\mathbf{v})^{1/2} = \alpha + \mathbf{B} \,, \tag{45}$$

where α is a scalar and $\mathbf{B}$ is a bivector, both to be determined by the constraints. Squaring both sides, we get

$$\mathbf{uv} = (\alpha^2 - |\,\mathbf{B}\,|^2) + 2\alpha\mathbf{B}\,. \tag{46}$$

This gives us

$$\mathbf{u}\cdot\mathbf{v} = \alpha^2 - |\,\mathbf{B}\,|^2\,, \tag{47a}$$
$$\mathbf{u}\wedge\mathbf{v} = 2\alpha\mathbf{B}\,. \tag{47b}$$

Now, we can determine a simple relation between α and $|\,\mathbf{B}\,|$:

$$1 = \mathbf{vuuv} = [(\alpha^2 - |\,\mathbf{B}\,|^2) - 2\alpha\mathbf{B}]\,[(\alpha^2 - |\,\mathbf{B}\,|^2) + 2\alpha\mathbf{B}]\,. \tag{48}$$

After multiplying this out and simplifying, we get that

$$1 = \alpha^2 + |\,\mathbf{B}\,|^2\,. \tag{49}$$

Next we add unity to both sides of (46):

$$1 + \mathbf{uv} = \alpha^2 + (1 - |\,\mathbf{B}\,|^2) + 2\alpha\mathbf{B} = 2\alpha^2 + 2\alpha\mathbf{B}\,. \tag{50}$$

Reorganizing this, we have

$$\alpha + \mathbf{B} = \frac{1 + \mathbf{uv}}{2\alpha}\,. \tag{51}$$

We're almost there. Now we need an expression for the α in the denominator. Eliminating $|\,\mathbf{B}\,|$ between (47a) and (49), and solving for α, gives us

$$\alpha = [\tfrac{1}{2}(1 + \mathbf{u}\cdot\mathbf{v})]^{1/2}\,. \tag{52}$$

Substituting this into the last equation, we get

$$\alpha + \mathbf{B} = \frac{1 + \mathbf{uv}}{[2(1 + \mathbf{u}\cdot\mathbf{v})]^{1/2}}\,, \tag{53}$$

which gives us the proof we needed for Eq. (43).

Returning to (43) and squaring both sides, we get

$$R^2 = \mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} = \cos a + i\hat{\mathbf{a}}\sin a\,. \tag{54}$$

Then

$$R = e^{\frac{1}{2}\mathbf{a}} = \cos\tfrac{1}{2}a + i\hat{\mathbf{a}}\sin\tfrac{1}{2}a = \frac{1 + \mathbf{uv}}{[2(1 + \mathbf{u}\cdot\mathbf{v})]^{1/2}} = \frac{1 + \cos a + i\hat{\mathbf{a}}\sin a}{[2(1 + \mathbf{u}\cdot\mathbf{v})]^{1/2}}\,, \tag{55}$$

On equating the scalar and bivector parts of this last equation, we have that

$$\cos\tfrac{1}{2}a = \frac{1 + \cos a}{[2(1 + \cos a)]^{1/2}}\,, \tag{56a}$$
$$\sin\tfrac{1}{2}a = \frac{\sin a}{[2(1 + \cos a)]^{1/2}}\,. \tag{56b}$$

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Problem (3.10) page 294 of NFCM. Given that a rotation $\mathcal{R}(\mathbf{x}) = R^\dagger \mathbf{x} R$ has the properties

$$\mathcal{R}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \mathbf{b}\,, \tag{57a}$$

$$\mathcal{R}(\mathbf{a}) = \mathbf{b}\,, \tag{57b}$$

show that

$$\pm R = \frac{1 + \mathbf{a}^{-1}\mathbf{b}}{[2(1 + \mathbf{a}^{-1} \cdot \mathbf{b})]^{1/2}}\,. \tag{58}$$

My plan is to just demonstrate that this spinor works. But first I need to show that the product $\mathbf{a}^{-1}\mathbf{b}$ is unimodular.

$$1 = (\mathbf{b}\mathbf{a}^{-1})(\mathbf{a}^{-1}\mathbf{b}) = a^{-2}b^2\,, \tag{59}$$

which implies that $a = b$, which we can also conclude from (57b). Furthermore, Eq. (57a) tells us that the plane of rotation is orthogonal to $\mathbf{a} \times \mathbf{b}$. An identity we'll soon need is that

$$\mathbf{q}\mathbf{p} = 2\mathbf{p} \cdot \mathbf{q} - \mathbf{p}\mathbf{q}\,. \tag{60}$$

Now, with the definition $K = [2(1 + \mathbf{a}^{-1} \cdot \mathbf{b})]^{1/2}$, then

$$\begin{aligned}
\mathcal{R}(\mathbf{a}) &= R^\dagger \mathbf{a} R \\
&= K^{-2}[\,1 + \mathbf{b}\mathbf{a}^{-1}\,]\,\mathbf{a}\,[\,1 + \mathbf{a}^{-1}\mathbf{b}\,] \\
&= K^{-2}[\,\mathbf{a} + \mathbf{b}\,]\,[\,1 + \mathbf{a}^{-1}\mathbf{b}\,] \\
&= K^{-2}[\,\mathbf{a} + 2\mathbf{b} + \mathbf{b}\mathbf{a}^{-1}\mathbf{b}\,] \\
&= K^{-2}[\,\mathbf{a} + 2\mathbf{b} + \mathbf{b}(2\mathbf{a}^{-1} \cdot \mathbf{b} - \mathbf{b}\mathbf{a}^{-1})\,] \\
&= K^{-2}[\,2(1 + \mathbf{a}^{-1} \cdot \mathbf{b})\mathbf{b})\,] \\
&= \mathbf{b}\,.
\end{aligned} \tag{61a}$$

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Problem (3.11) page 294 of NFCM. For the composition of rotations given by the spinor equation

$$R_1 R_2 = R_3\,, \tag{62}$$

where

$$R_k = e^{\frac{1}{2} i \mathbf{a}_k} = \alpha_k (1 + i\boldsymbol{\gamma}_k)\,, \tag{63}$$

derive the Law of Tangents:

$$\tan \tfrac{1}{2}\mathbf{a}_3 = \boldsymbol{\gamma}_3 = \frac{\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 + \boldsymbol{\gamma}_2 \times \boldsymbol{\gamma}_1}{1 - \boldsymbol{\gamma}_1 \cdot \boldsymbol{\gamma}_2}\,. \tag{64}$$

$$\begin{aligned} R_1 R_2 &= \alpha_1(1+i\boldsymbol{\gamma}_1)\alpha_2(1+i\boldsymbol{\gamma}_2) \\ &= \alpha_1\alpha_2[\,1+i(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)-\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\,]\,, \end{aligned} \tag{65}$$

But

$$R_3 = \alpha_3(1+i\boldsymbol{\gamma}_3) = \alpha_1\alpha_2[\,1+i(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)-\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\,] = e^{\frac{1}{2}i\mathbf{a}}\,. \tag{66}$$

On equating the scalar and bivector parts, we get:

$$\text{Scalar Part:} \qquad \alpha_3 = \alpha_1\alpha_2(1-\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2) = \cos\tfrac{1}{2}\mathbf{a}\,, \tag{67a}$$

$$\text{Bivector Part:} \quad \alpha_3\boldsymbol{\gamma}_3 i = \alpha_1\alpha_2[\,i(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)-\boldsymbol{\gamma}_1\wedge\boldsymbol{\gamma}_2\,] = i\sin\tfrac{1}{2}\mathbf{a}\,. \tag{67b}$$

Taking the dual of this last equation, we get

$$\alpha_3\boldsymbol{\gamma}_3 = \alpha_1\alpha_2[\,(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)+\boldsymbol{\gamma}_2\times\boldsymbol{\gamma}_2\,] = \sin\tfrac{1}{2}\mathbf{a}\,. \tag{68}$$

Now,

$$\tan\tfrac{1}{2}\mathbf{a} = \frac{\sin\frac{1}{2}\mathbf{a}}{\cos\frac{1}{2}\mathbf{a}} = \frac{\alpha_3\boldsymbol{\gamma}_3}{\alpha_3} = \boldsymbol{\gamma}_3\,. \tag{69}$$

Using (67a) and (68), we have that

$$\tan\tfrac{1}{2}\mathbf{a} = \boldsymbol{\gamma}_3 = \frac{\alpha_1\alpha_2[\,(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)+\boldsymbol{\gamma}_2\times\boldsymbol{\gamma}_2\,]}{\alpha_1\alpha_2(1-\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2)} = \frac{\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2+\boldsymbol{\gamma}_2\times\boldsymbol{\gamma}_2}{1-\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2}\,. \tag{70}$$

♣

Problem (3.12) page 294 of NFCM. The sum of the diagonal matrix elements f_{kk} of a linear transformation f is called the *trace* of f and denoted by Tr f. Show that the trace of a rotation $\mathcal{R}$ is given by

$$\text{Tr }\mathcal{R} = \sum_k \boldsymbol{\sigma}_k\cdot(\mathcal{R}\boldsymbol{\sigma}_k) = 1+2\cos\mathbf{a}\,, \tag{71}$$

where $\mathbf{a}$ is the vector angle of rotation.

It's best to consider this problem as a corollary to problem (3.7). If we remember how we derived the equation

$$e_{jk} = \delta_{jk}\cos a - \epsilon_{jkm}\hat{a}_m\sin a + \hat{a}_j\hat{a}_k(1-\cos a)\,, \tag{72}$$

then the quantity $\sum_k \boldsymbol{\sigma}_k\cdot(\mathcal{R}\boldsymbol{\sigma}_k)$ is just the value of $\sum_k e_{kk}$. Therefore,

$$\begin{aligned} \text{Tr }\mathcal{R} = \sum_k e_{kk} &= \sum_k [\,\delta_{kk}\cos a - \epsilon_{kkm}\hat{a}_m\sin a + \hat{a}_k\hat{a}_k(1-\cos a)\,] \\ &= 3\cos a + \sum_k \hat{a}_k\hat{a}_k(1-\cos a)\,, \end{aligned} \tag{73}$$

where we used that 1) $\sum_k \delta_{kk} = 3$ and that 2) for every k, $\epsilon_{kkm} = 0$. Lastly, since the $\hat{a}_k$'s are direction cosines, then

$$\sum_k \hat{a}_k \hat{a}_k = \hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2 = 1\,. \tag{74}$$

On substituting this last result into (73), we get

$$\mathrm{Tr}\ \mathcal{R} = \sum_k \boldsymbol{\sigma}_k \cdot (\mathcal{R}\boldsymbol{\sigma}_k) = 1 + 2\cos\mathbf{a}\,, \tag{75}$$

where $\cos\mathbf{a} = \cos a$.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.