Notes on Chapter 5, Section 3

P. Reany

January 6, 2022

1 Reflections and Rotations

These notes cover pages 277 to 295 of NFCM [1].

I begin with the subsection: Matrix Elements of a Rotation.

Let a standard frame be given by $\{\sigma_k\}$ $(k = 1, 2, 3)$. Then, for each k,

$$\mathbf{e}_k = R^\dagger \sigma_k R = \sum_j \sigma_j \, e_{jk} \,, \tag{1}$$

where R is the spinor of the rotation. Then,

$$\begin{aligned} \mathbf{e}_k \cdot \sigma_\ell &= \sum_j \sigma_j \cdot \sigma_\ell \, e_{jk} = e_{\ell k} \\ &= \langle \, R^\dagger \sigma_k R \sigma_\ell \, \rangle \,. \end{aligned} \tag{2}$$

If we now parameterize R with the Euler parameters, we have

$$R = \alpha + i\boldsymbol{\beta} \,, \tag{3}$$

and (2) becomes

$$\begin{aligned} \mathbf{e}_k \cdot \sigma_\ell &= \langle \, (\alpha - i\boldsymbol{\beta})\sigma_k(\alpha + i\boldsymbol{\beta})\sigma_\ell \, \rangle \\ &= \langle \, (\alpha\sigma_k - i\boldsymbol{\beta}\sigma_k)(\alpha\sigma_\ell + i\boldsymbol{\beta}\sigma_\ell) \, \rangle \\ &= \langle \, \alpha^2 \sigma_k \sigma_\ell + \alpha i \sigma_k \boldsymbol{\beta}\sigma_\ell - \alpha i \beta \sigma_k \sigma_\ell + \boldsymbol{\beta}\sigma_k \boldsymbol{\beta}\sigma_\ell \, \rangle \\ &= \alpha^2 \sigma_k \cdot \sigma_\ell + \alpha \langle \, i\boldsymbol{\beta}(\sigma_\ell \sigma_k - \sigma_k \sigma_\ell) \, \rangle + \langle \, \boldsymbol{\beta}\sigma_k \boldsymbol{\beta}\sigma_\ell \, \rangle \\ &= \alpha^2 \delta_{k\ell} + 2\alpha \langle \, i\boldsymbol{\beta} \, \sigma_\ell \wedge \sigma_k \, \rangle + \langle \, \boldsymbol{\beta}\sigma_k \boldsymbol{\beta}\sigma_\ell \, \rangle \,. \end{aligned} \tag{4}$$

Now, using that

$$\epsilon_{ijk} = -\langle \, i\sigma_i \sigma_j \sigma_k \, \rangle = \sigma_i \cdot \sigma_j \times \sigma_k \,, \tag{5}$$

by expanding $\boldsymbol{\beta}$ in a basis, $\langle \, i\boldsymbol{\beta} \, \sigma_\ell \wedge \sigma_k \, \rangle$ becomes

$$\begin{aligned} \langle \, i\boldsymbol{\beta} \, \sigma_\ell \wedge \sigma_k \, \rangle &= \langle \, i\Big(\sum_j \beta_j \sigma_j\Big) \wedge \sigma_\ell \wedge \sigma_k \, \rangle \\ &= \sum_j \beta_j \langle \, i\sigma_j \wedge \sigma_\ell \wedge \sigma_k \, \rangle = -\sum_j \beta_j \epsilon_{j\ell k} \,. \end{aligned} \tag{6}$$

Then (4) becomes

$$e_{k\ell} = \alpha^2\delta_{k\ell} - 2\alpha\sum_j \beta_j\epsilon_{j\ell k} + \langle\, \boldsymbol{\beta}\sigma_k\boldsymbol{\beta}\sigma_\ell\,\rangle\,. \tag{7}$$

We can now simplify the last term by using that $\boldsymbol{\beta}\sigma_k = 2\boldsymbol{\beta}\cdot\sigma_k - \sigma_k\boldsymbol{\beta}$:

$$\begin{aligned}\langle\, \boldsymbol{\beta}\sigma_k\boldsymbol{\beta}\sigma_\ell\,\rangle &= \langle\, (2\boldsymbol{\beta}\cdot\sigma_k - \sigma_k\boldsymbol{\beta})\boldsymbol{\beta}\sigma_\ell\,\rangle \\ &= 2\boldsymbol{\beta}\cdot\sigma_k\langle\, \boldsymbol{\beta}\sigma_\ell\,\rangle - \langle\, \sigma_k\boldsymbol{\beta}^2\sigma_\ell\,\rangle \\ &= 2\beta_k\beta_\ell - \beta^2\delta_{k\ell}\,. \end{aligned} \tag{8}$$

And, then, (7) becomes

$$e_{k\ell} = \alpha^2\delta_{k\ell} - 2\alpha\sum_j \beta_j\epsilon_{j\ell k} + 2\beta_k\beta_\ell - \beta^2\delta_{k\ell}\,. \tag{9}$$

We have one more simplification that arises from the fact that $\beta^2 = 1-\alpha^2$:

$$e_{k\ell} = (2\alpha^2-1)\delta_{k\ell} + 2\beta_k\beta_\ell - 2\alpha\sum_j \beta_j\epsilon_{j\ell k}\,, \tag{10}$$

which is Eq. (3.33) on page 286 of the text.

Page 287:

Our next task is to solve for R in terms of $\mathbf{e}_k$ and σ_k. To accomplish this we introduce the quaternion

$$T \equiv \sum_k \sigma_k\mathbf{e}_k = \sum_{j,k} e_{jk}\sigma_k\sigma_j\,. \tag{11}$$

Our game plan now is to relate T to the spinor R so as to solve for R as a function of T.

We can rewrite (11) as

$$T = \sum_k \sigma_k R^\dagger\sigma_k R\,. \tag{12}$$

Now we use that $R^\dagger = \alpha + i\beta$ and that $\sum_k \sigma_k\boldsymbol{\beta}\sigma_k = -\boldsymbol{\beta}$, to get

$$\begin{aligned}T &= \sum_k \sigma_k(\alpha - i\boldsymbol{\beta})\sigma_k R = \alpha\sum_k \sigma_k\sigma_k\, R - i\sum_k \sigma_k\boldsymbol{\beta}\sigma_k R \\ &= (3\alpha + i\boldsymbol{\beta})R = (3\alpha + [\alpha - R^\dagger])R \\ &= 4\alpha R - 1\,. \end{aligned} \tag{13}$$

On reorganizing, we get

$$4\alpha R = 1 + T\,. \tag{14}$$

On taking the scalar part of this last equation, we arrive at

$$4\alpha^2 = \langle\, 1 + T\, \rangle\,. \tag{15}$$

Then, on multiplying (17) through by its reverse, we get that

$$16\alpha^2 = |\, 1 + T\, |^2\,. \tag{16}$$

Solving (17) for R, we get

$$\pm R = \frac{1 + T}{2\langle\, 1 + T\, \rangle^{1/2}} = \frac{1 + T}{|\, 1 + T\, |}\,. \tag{17}$$

♣

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.