Problems from Chapter 5, Section 4

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1 Transformation Groups

Problem (4.1) page 306 of NFCM [1].

Prove that the translations satisfy each of the four group properties.

A group is 1) a set of elements with a closed binary operation. 2) Every element of the group has an inverse. 3) There exists an identity element of the group. 4) The group operation is associative.

To prove 1) we need to show that for arbitrary translations $\mathcal{T}_{\mathbf{a}}$ and $\mathcal{T}_{\mathbf{b}}$

$$\mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}} = \mathcal{T}_{\mathbf{c}}\,, \tag{1}$$

where $\mathbf{c}$ is a vector in the space. Actually, $\mathbf{c} = \mathbf{a} + \mathbf{b}$, which we'll show now.

$$\mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}}\mathbf{x} = \mathcal{T}_{\mathbf{a}}(\mathbf{x} + \mathbf{b}) = \mathbf{x} + \mathbf{b} + \mathbf{a} = \mathbf{x} + \mathbf{a} + \mathbf{b} = \mathcal{T}_{\mathbf{a}+\mathbf{b}}\mathbf{x}\,. \tag{2}$$

I leave it to the reader to show that multiplication of translations is commutative. Groups don't have to be commutative, but this one is.

To prove 2) we need to show that for arbitrary translation $\mathcal{T}_{\mathbf{a}}$ there exists a translation $\mathcal{T}_{\mathbf{b}}$, such that

$$\mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}}\mathbf{x} = \mathbf{x}\,. \tag{3}$$

where $\mathbf{c}$ is a vector in the space. Actually, $\mathbf{c} = \mathbf{a} + \mathbf{b}$, which we'll show now.

$$\mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}}\mathbf{x} = \mathcal{T}_{\mathbf{a}}(\mathbf{x} + \mathbf{b}) = \mathbf{x} + \mathbf{b} + \mathbf{a} = \mathbf{x} + \mathbf{a} + \mathbf{b} = \mathcal{T}_{\mathbf{a}+\mathbf{b}}\mathbf{x}\,, \tag{4}$$

From (2) we see that $\mathbf{b} = -\mathbf{a}$.

Now for 3). This brings us to the existence of an identity translation, which clearly has to be $\mathcal{T}_{\mathbf{0}}$.

Finally, we need to show that the product of translations is associative. The proof will boil down to using the fact that vector addition in a vector space is associative.

$$\begin{aligned}(\mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}})\mathcal{T}_{\mathbf{c}}\mathbf{x} &= \mathcal{T}_{\mathbf{a}+\mathbf{b}}\mathcal{T}_{\mathbf{c}}\mathbf{x}\\ &= \mathcal{T}_{\mathbf{a}+\mathbf{b}+\mathbf{c}}\mathbf{x}\\ &= \mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}+\mathbf{c}}\mathbf{x}\\ &= \mathcal{T}_{\mathbf{a}}(\mathcal{T}_{\mathbf{b}}\mathcal{T}_{\mathbf{c}})\mathbf{x}\,. \end{aligned} \tag{5}$$

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Problem (4.2) Derive (4.25) and prove that the isometries of Euclidean space form a group.

1) Proof that the composition of isometries is closed.

Given that an isometry $\mathcal{R}$ can be expressed as

$$\{R|\mathbf{a}\}\mathbf{x} = \widetilde{R}\mathbf{x}R + \mathbf{a}\,, \tag{6}$$

show that successive isometries $\mathcal{SR}$ can be expressed as

$$\{S|\mathbf{b}\}\{R|\mathbf{a}\} = \{RS|\widetilde{S}\mathbf{a}S + \mathbf{b}\}\,. \tag{7}$$

Proof:

$$\begin{aligned}\{S|\mathbf{b}\}\{R|\mathbf{a}\}\mathbf{x} &= \{S|\mathbf{b}\}[\,\widetilde{R}\mathbf{x}R + \mathbf{a}\,] \\ &= \widetilde{S}[\,\widetilde{R}\mathbf{x}R + \mathbf{a}\,]S + \mathbf{b} \\ &= (\widetilde{S}\widetilde{R})\mathbf{x}RS + \widetilde{S}\mathbf{a}\,]S + \mathbf{b} \\ &= (RS)^{\sim}\mathbf{x}RS + \widetilde{S}\mathbf{a}S + \mathbf{b} \\ &= \{RS|\widetilde{S}\mathbf{a}S + \mathbf{b}\}\mathbf{x}\,. \end{aligned} \tag{8}$$

2) Proof that isometries have inverses.

$$\{R|\mathbf{a}\}^{-1} = \{\widetilde{R}| - R\mathbf{a}\widetilde{R}\}\,. \tag{9}$$

We begin with the assignment

$$\{R|\mathbf{a}\}^{-1} = \{S|\mathbf{b}\}\,, \tag{10}$$

where we need to solve for S and $\mathbf{b}$. Thus

$$\{S|\mathbf{b}\}\{R|\mathbf{a}\}\mathbf{x} = \{RS|\widetilde{S}\mathbf{a}S + \mathbf{b}\}\mathbf{x} = \mathbf{x}\,, \tag{11}$$

or

$$(RS)^{\sim}\mathbf{x}(RS) + \widetilde{S}\mathbf{a}S + \mathbf{b} = \mathbf{x}\,, \tag{12}$$

with solution $S = \widetilde{R}$ and $\widetilde{S}\mathbf{a}S + \mathbf{b} = 0$, or

$$\mathbf{b} = -R\mathbf{a}\widetilde{R}\,. \tag{13}$$

Hence,

$$\{R|\mathbf{a}\}^{-1} = \{\widetilde{R}| - R\mathbf{a}\widetilde{R}\}\,. \tag{14}$$

But wait! Inverses must be two-sided. We have yet to show that

$$\{R|\mathbf{a}\}\{S|\mathbf{b}\}\mathbf{x} = \mathbf{x}\,, \tag{15}$$

which I'll leave to the reader.

3) It's obvious that the identity isometry is $\{1|\mathbf{0}\}$.

4) Now for the fun one: Associativity! We need to show that for arbitrary isometries $\{R|\mathbf{a}\}, \{S|\mathbf{b}\}, \{T|\mathbf{c}\}$,

$$\big[\,\{R|\mathbf{a}\}\{S|\mathbf{b}\}\,\big]\{T|\mathbf{c}\}\mathbf{x} = \{R|\mathbf{a}\}\big[\,\{S|\mathbf{b}\}\{T|\mathbf{c}\}\,\big]\mathbf{x}\,. \tag{16}$$

Let's make some convenient substitutions at this point:

$$\begin{aligned} \{R|\mathbf{a}\}\{S|\mathbf{b}\} &= \{N|\mathbf{d}\} \quad \text{where} \quad N = SR\,,\ \mathbf{d} = \widetilde{R}\mathbf{b}R + \mathbf{a}\,, \\ \{S|\mathbf{b}\}\{T|\mathbf{c}\} &= \{M|\mathbf{e}\} \quad \text{where} \quad M = TS\,,\ \mathbf{e} = \widetilde{S}\mathbf{c}S + \mathbf{b}\,. \end{aligned} \tag{17}$$

Thus, the LHS of (16) becomes

$$\begin{aligned} \big[\,\{R|\mathbf{a}\}\{S|\mathbf{b}\}\,\big]\{T|\mathbf{c}\}\mathbf{x} &= \{N|\mathbf{d}\}\{T|\mathbf{c}\}\mathbf{x} \\ &= \{N|\mathbf{d}\}(\widetilde{T}\mathbf{x}T + \mathbf{c}) \\ &= \widetilde{N}(\widetilde{T}\mathbf{x}T + \mathbf{c})N + \mathbf{d} \\ &= \widetilde{N}\widetilde{T}\mathbf{x}TN + \widetilde{N}\mathbf{c}N + \mathbf{d} \\ &= (SR)^{\sim}\widetilde{T}\mathbf{x}T(SR) + (SR)^{\sim}\mathbf{c}(SR) + \mathbf{d} \\ &= (TSR)^{\sim}\mathbf{x}(TSR) + (SR)^{\sim}\mathbf{c}(SR) + \widetilde{R}\mathbf{b}R + \mathbf{a}\,. \end{aligned} \tag{18}$$

And now the RHS of (16) becomes

$$\begin{aligned} \{R|\mathbf{a}\}\big[\,\{S|\mathbf{b}\}\{T|\mathbf{c}\}\,\big]\mathbf{x} &= \{R|\mathbf{a}\}\{M|\mathbf{e}\}\mathbf{x} \\ &= \{R|\mathbf{a}\}(\widetilde{M}\mathbf{x}M + \mathbf{e}) \\ &= \widetilde{R}(\widetilde{M}\mathbf{x}M + \mathbf{e})R + \mathbf{a} \\ &= \widetilde{R}\widetilde{M}\mathbf{x}MR + \widetilde{R}\mathbf{e}R + \mathbf{a} \\ &= \widetilde{R}(TS)^{\sim}\mathbf{x}(TS)R + \widetilde{R}(\widetilde{S}\mathbf{c}S + \mathbf{b})R + \mathbf{a} \\ &= (TSR)^{\sim}\mathbf{x}(TSR) + (SR)^{\sim}\mathbf{c}(SR) + \widetilde{R}\mathbf{b}R + \mathbf{a}\,. \end{aligned} \tag{19}$$

By comparing the results of (18) and (19), it's clear that we have established (16). ♣

Problem (4.3) Prove that any rigid displacement with a fixed point is a rotation.

The proof is left to the reader.

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Problem (4.4) Prove that rotations with parallel axes do not generally commute unless the axes coincide.

The proof is left to the reader.

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Problem (4.5) Derive Eq. (4.34) from Eq. (4.33).

$$\mathbf{p}_\perp = \mathbf{a}_\perp (1 - R^2)^{-1} \,. \tag{20}$$

The rest of this demonstration is little more than just manipulating complex numbers and some trigonometry. According to the text, we assume that R has the form

$$R = e^{\frac{1}{2}i\mathbf{n}\phi} \,. \tag{21}$$

Then

$$R^2 = e^{i\mathbf{n}\phi} \,. \tag{22}$$

Thus,

$$\begin{aligned}
\mathbf{b}_\perp &= \mathbf{a}_\perp (1 - e^{i\mathbf{n}\phi})^{-1} \\
&= \mathbf{a}_\perp \frac{(1 - e^{-i\mathbf{n}\phi})}{(1 - e^{i\mathbf{n}\phi})(1 - e^{-i\mathbf{n}\phi})} \\
&= \mathbf{a}_\perp \frac{1 - e^{-i\mathbf{n}\phi}}{1 - (e^{-i\mathbf{n}\phi} + e^{i\mathbf{n}\phi}) + 1} \\
&= \tfrac{1}{2}\mathbf{a}_\perp \frac{1 - e^{-i\mathbf{n}\phi}}{1 - \cos\phi} \\
&= \tfrac{1}{2}\mathbf{a}_\perp \frac{1 - (\cos\phi - i\mathbf{n}\sin\phi)}{1 - \cos\phi} \\
&= \tfrac{1}{2}\mathbf{a}_\perp \left[1 + i\mathbf{n}\frac{\sin\phi}{1 - \cos\phi}\right] \\
&= \tfrac{1}{2}\mathbf{a}_\perp \left[1 + i\mathbf{n}\cot\tfrac{1}{2}\phi\right] \\
&= \tfrac{1}{2}\left[\mathbf{a}_\perp + \mathbf{n} \times \mathbf{a}\cot\tfrac{1}{2}\phi\right] .
\end{aligned} \tag{23}$$

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Problem (4.6) A rigid displacement $\{R|\mathbf{a}\}$ can be expressed as the product of a translation $\mathcal{T}_\mathbf{c}$ and a rotation $\mathcal{R}_\mathbf{b}$ centered at point $\mathbf{b}$, i.e.,

$$\{R|\mathbf{a}\} = \mathcal{T}_\mathbf{c}\mathcal{R}_\mathbf{b} \,. \tag{24}$$

Determine the translation vector $\mathbf{c}$.

Solution:

$$
\begin{aligned}
\{R|\mathbf{a}\}\mathbf{x} &= \mathcal{T}_{\mathbf{c}}\mathcal{R}_{\mathbf{b}}\mathbf{x}\,, \\
\widetilde{R}\mathbf{x}R + \mathbf{a} &= \mathcal{T}_{\mathbf{c}}[\,\widetilde{R}\mathbf{x}R + \mathbf{b} - \widetilde{R}\mathbf{b}R\,] \\
&= \widetilde{R}\mathbf{x}R + \mathbf{b} - \widetilde{R}\mathbf{b}R + \mathbf{c}\,.
\end{aligned} \tag{25}
$$

Solving for $\mathbf{c}$, we get

$$
\mathbf{c} = \mathbf{a} - \mathbf{b} + \widetilde{R}\mathbf{b}R\,. \tag{26}
$$

♣

Problem (4.7) A subgroup $\{\mathcal{T}\}$ of a group $\{\mathcal{G}\}$ is said to be an invariant subgroup if

$$
\mathcal{G}^{-1}\mathcal{T}\mathcal{G} \in \{\mathcal{T}\} \text{ for all } \mathcal{T} \in \{\mathcal{T}\} \text{ and for all } \mathcal{G} \in \{\mathcal{G}\}\,. \tag{27}
$$

Prove that the translations comprise an invariant subgroup of the Euclidean Isometry group.[1]

Proof:

We'll write our basic isometry as $\{R|\mathbf{a}\}$ for some arbitrary vector $\mathbf{a}$ and rotation R. Now, we'll let $\mathcal{T}_{\mathbf{c}}$ represent an arbitrary translation by the vector $\mathbf{c}$. Thus, employing Eq. (14) for the inverse of an isometry, (27) becomes

$$
\begin{aligned}
\{\widetilde{R}| - R\mathbf{a}\widetilde{R}\}\mathcal{T}_{\mathbf{c}}\{R|\mathbf{a}\}\mathbf{x} &= \{\widetilde{R}| - R\mathbf{a}\widetilde{R}\}\mathcal{T}_{\mathbf{c}}(\widetilde{R}\mathbf{x}R + \mathbf{a}) \\
&= \{\widetilde{R}| - R\mathbf{a}\widetilde{R}\}(\widetilde{R}\mathbf{x}R + \mathbf{a} + \mathbf{c}) \\
&= R(\widetilde{R}\mathbf{x}R + \mathbf{a} + \mathbf{c})\widetilde{R} + (-R\mathbf{a}\widetilde{R}) \\
&= \mathbf{x} + R\mathbf{c}\widetilde{R} \\
&= \mathcal{T}_{R\mathbf{c}\widetilde{R}}\mathbf{x}\,.
\end{aligned} \tag{28}
$$

Hence, conjugating a translation by a generic isometry is still a translation.

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Problem (4.8) Let $\mathcal{S}$ denote the reflection along a (nonzero) vector $\mathbf{a}$. If $\mathcal{T}_{\mathbf{a}}$ is the translation by $\mathbf{a}$, then

$$
\mathcal{S}_{\mathbf{a}} = \mathcal{T}_{\mathbf{a}}\mathcal{S}\mathcal{T}_{\mathbf{a}}^{-1} \tag{29}
$$

is the reflection $\mathcal{S}$ shifted to the point $\mathbf{a}$. Show that

$$
\mathcal{S}\mathcal{S}_{(-\mathbf{a})} = \mathcal{T}_{2\mathbf{a}}\,. \tag{30}
$$

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[1] This construction is also known as 'conjugation' of a group element a. Conjugation is given in either of the following forms: Either $g^{-1}ag$ or gag^{-1}, as one includes all elements g from some group G, say. Since every element of a group has an inverse, it doesn't matter which side of the conjugated element (in this case a) we put the inverse of g, though consistency in a given problem or lecture is required. See also *normal subgroup*.

If you switch the order of $\mathcal{G}$ and $\mathcal{G}^{-1}$ around $\mathcal{T}$ in (27), you will, in general, obtain a different translation, but it's still within the translation subgroup, which is all that matters.

Thus, a translation by $\mathbf{a}$ can be expressed as a product of reflections in parallel planes separated by a distance $\frac{1}{2}\mathbf{a}$.

The solution is left to the reader.
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References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.