Notes on Chapter 5, Section 4

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February 28, 2022

1 Transformation Groups

These notes cover pages 295 to 306 of NFCM [1].

We begin on page 301 for the Euclidean group. Given that an isometry can be expressed as

$$\mathbf{x}' = f(\mathbf{x})\,, \tag{1}$$

with the property that

$$(\mathbf{x}' - \mathbf{y}')^2 = (\mathbf{x} - \mathbf{y})^2\,. \tag{2}$$

Then

$$\mathbf{x}' - \mathbf{y}' = f(\mathbf{x}) - f(\mathbf{y}) = \mathcal{R}(\mathbf{x} - \mathbf{y})\,. \tag{3a}$$

This equation is true for the pair $\mathbf{y}$ and $\mathbf{z}$, was well, yielding

$$\mathbf{y}' - \mathbf{z}' = \mathcal{R}(\mathbf{y} - \mathbf{z})\,. \tag{3b}$$

Adding these last two equations, we get

$$\mathbf{x}' - \mathbf{z}' = \mathcal{R}(\mathbf{x} - \mathbf{y}) + \mathcal{R}(\mathbf{y} - \mathbf{z})\,. \tag{4}$$

But we also have that

$$\mathbf{x}' - \mathbf{z}' = \mathcal{R}(\mathbf{x} - \mathbf{z})\,. \tag{5}$$

Hence

$$\mathcal{R}(\mathbf{x} - \mathbf{z}) = \mathcal{R}(\mathbf{x} - \mathbf{y}) + \mathcal{R}(\mathbf{y} - \mathbf{z})\,. \tag{6}$$

On setting $\mathbf{y}$ to zero in (6), we have that

$$\mathcal{R}(\mathbf{x} - \mathbf{z}) = \mathcal{R}(\mathbf{x}) + \mathcal{R}(-\mathbf{z})\,. \tag{7}$$

On setting $\mathbf{z} = -\mathbf{x}$ in this last equation:

$$\mathcal{R}(2\mathbf{x}) = 2\mathcal{R}(\mathbf{x})\,. \tag{8}$$

Now, we use induction to show that

$$\mathcal{R}(m\mathbf{x}) = m\mathcal{R}(\mathbf{x})\,, \tag{9}$$

for any natural number m. Inductive step: Assume that

$$\mathcal{R}((k-1)\mathbf{x}) = (k-1)\mathcal{R}(\mathbf{x}), \tag{10}$$

for any natural numbers $k > 1$, then, letting $\mathbf{z} = -(k-1)\mathbf{x}$ in (7), we get

$$\begin{aligned} \mathcal{R}(k\mathbf{x}) &= \mathcal{R}(\mathbf{x}) + \mathcal{R}((k-1)\mathbf{x}) \\ &= \mathcal{R}(\mathbf{x}) + (k-1)\mathcal{R}(\mathbf{x}) \\ &= k\mathcal{R}(\mathbf{x}), \end{aligned} \tag{11}$$

which is what we were to show.

To show that

$$\mathcal{R}(\alpha\mathbf{x}) = \alpha\mathcal{R}(\mathbf{x}), \tag{12}$$

for arbitrary scalar α (a real number), we first show that (12) is true for any rational number given as the ratio of two integers m, n (for the moment positive). Multiply (9) through by n to get

$$n\mathcal{R}(m\mathbf{x}) = m[n\mathcal{R}(\mathbf{x})] = m\mathcal{R}(n\mathbf{x}). \tag{13}$$

Now, replace $\mathbf{x}$ by $\mathbf{y}/n$:

$$n\mathcal{R}\big(\frac{m}{n}\mathbf{y}\big) = m\mathcal{R}(\mathbf{y}). \tag{14}$$

On dividing through by n, we get

$$\mathcal{R}\big(\frac{m}{n}\mathbf{y}\big) = \frac{m}{n}\mathcal{R}(\mathbf{y}). \tag{15}$$

Since every positive real number can be approximated arbitrarily closely by some rational number, we accept (12) as correct for α any positive real number.

We can show that

$$\mathcal{R}(\mathbf{0}) = 0. \tag{16}$$

From (3a), we get that

$$(\mathbf{x}-\mathbf{y})^2 = [\mathcal{R}(\mathbf{x}-\mathbf{y})]^2. \tag{17}$$

Setting $\mathbf{x}-\mathbf{y} = \mathbf{0}$, gives us

$$(\mathbf{0})^2 = [\mathcal{R}(\mathbf{0})]^2. \tag{18}$$

Since our algebra doesn't admit zero-divisors, then (16) must be true.

To extend (12) to all real numbers, it suffices to show that

$$\mathcal{R}(-\mathbf{x}) = -\mathcal{R}(\mathbf{x}). \tag{19}$$

To that end, let's set $\mathbf{z}$ to $\mathbf{x}$ in (6), to get

$$\mathcal{R}(\mathbf{0}) = 0 = \mathcal{R}(\mathbf{x}-\mathbf{y}) + \mathcal{R}(\mathbf{y}-\mathbf{x}). \tag{20}$$

On setting $\mathbf{x}-\mathbf{y}$ to $\mathbf{a}$, we get

$$0 = \mathcal{R}(\mathbf{a}) + \mathcal{R}(-\mathbf{a}), \tag{21}$$

where $\mathbf{a}$ is arbitrary. Therefore,

$$\mathcal{R}(-\mathbf{a}) = -\mathcal{R}(\mathbf{a})\,. \tag{22}$$

♣

We next go to page 302. Given that an isometry $\mathcal{R}$ can be expressed as

$$\{R|\mathbf{a}\}\mathbf{x} = \widetilde{R}\mathbf{x}R + \mathbf{a}\,, \tag{23}$$

show that successive isometries $\mathcal{S}\mathcal{R}$ can be expressed as

$$\{S|\mathbf{b}\}\{R|\mathbf{a}\} = \{RS|\widetilde{S}\mathbf{a}S + \mathbf{b}\}\,. \tag{24}$$

Proof:

$$\begin{aligned}\{S|\mathbf{b}\}\{R|\mathbf{a}\}\mathbf{x} &= \{S|\mathbf{b}\}[\,\widetilde{R}\mathbf{x}R + \mathbf{a}\,] \\ &= \widetilde{S}[\,\widetilde{R}\mathbf{x}R + \mathbf{a}\,]S + \mathbf{b} \\ &= (\widetilde{S}\widetilde{R})\mathbf{x}RS + \widetilde{S}\mathbf{a}\,]S + \mathbf{b} \\ &= (RS)^{\sim}\mathbf{x}RS + \widetilde{S}\mathbf{a}S + \mathbf{b} \\ &= \{RS|\widetilde{S}\mathbf{a}S + \mathbf{b}\}\,. \end{aligned} \tag{25}$$

We're now well set up to prove that

$$\{R|\mathbf{a}\}^{-1} = \{\widetilde{R}| - R\mathbf{a}\widetilde{R}\}\,, \tag{26}$$

We begin with the assignment

$$\{R|\mathbf{a}\}^{-1} = \{S|\mathbf{b}\}\,, \tag{27}$$

where we need to solve for S and $\mathbf{b}$. Thus

$$\{S|\mathbf{b}\}\{R|\mathbf{a}\}\mathbf{x} = \{RS|\widetilde{S}\mathbf{a}S + \mathbf{b}\}\mathbf{x} = \mathbf{x}\,, \tag{28}$$

or

$$(RS)^{\sim}\mathbf{x}(RS) + \widetilde{S}\mathbf{a}S + \mathbf{b} = \mathbf{x}\,, \tag{29}$$

with solution $S = \widetilde{R}$ and $\widetilde{S}\mathbf{a}S + \mathbf{b} = 0$, or

$$\mathbf{b} = -R\mathbf{a}\widetilde{R}\,. \tag{30}$$

But wait! Inverses must be two-sided. We have yet to show that

$$\{R|\mathbf{a}\}\{S|\mathbf{b}\}\mathbf{x} = \mathbf{x}\,, \tag{31}$$

which I'll leave to the reader.

p. 303

Now, to calculate the general form for a rotation about an arbitrary point $\mathbf{b}$:

First, translate the point $\mathbf{b}$ to the origin by $\mathcal{T}_{-\mathbf{b}}\,\mathbf{x}$. Of course, this will translate the whole space as well. Second, do the rotation about the origin with $\mathcal{R}$. Lastly, translate the entire space in the direction of $\mathbf{b}$ with the operator $\mathcal{T}_{\mathbf{b}}$, which will restore the point $\mathbf{b}$ back to where it started. Hence,

$$\begin{aligned}
\mathcal{R}_{\mathbf{b}}\mathbf{x} &= \mathcal{T}_{\mathbf{b}}\mathcal{R}\mathcal{T}_{-\mathbf{b}}\,\mathbf{x} \\
&= \mathcal{T}_{\mathbf{b}}\mathcal{R}(\mathbf{x}-\mathbf{b}) \\
&= \mathcal{T}_{\mathbf{b}}\widetilde{R}(\mathbf{x}-\mathbf{b})R \\
&= \widetilde{R}(\mathbf{x}-\mathbf{b})R+\mathbf{b} \\
&= \widetilde{R}\mathbf{x}R-\widetilde{R}\mathbf{b}R+\mathbf{b} \\
&= \{R|\mathbf{b}-\widetilde{R}\mathbf{b}R\}\mathbf{x}\,. \qquad (32)
\end{aligned}$$

Therefore,

$$\mathcal{R}_{\mathbf{b}} = \{R|\mathbf{b}-\widetilde{R}\mathbf{b}R\}\,. \qquad (33)$$

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Our next task is to go from Eq. (4.33) to Eq. (4.34) in the text. Eq. (4.33) is

$$\mathbf{b}_\perp = \mathbf{a}_\perp(1-R^2)^{-1}\,. \qquad (34)$$

We're interested in the points that are fixed during a rigid displacement $\{\mathcal{R}|\mathbf{a}\}\mathbf{x}$.

$$\{\mathcal{R}|\mathbf{a}\}\mathbf{x} = \mathbf{x}\,. \qquad (35)$$

We want to know the points $\{\mathbf{p}\}$ in 3d that satisfy the last equation, so that

$$R^\dagger\mathbf{p}R+\mathbf{a} = \mathbf{p}\,. \qquad (36)$$

My first comment is that the vector $\mathbf{a}$ has to be rather special. Let $\mathbf{n}$ point along the direction of the rotation axis. So, we have foliated space into an infinite family of parallel planes, each orthogonal to $\mathbf{n}$. We don't really care at this point in the analysis, the point in a given plane about whch the plane rotates. But to demand that the displacement is not also a translation, we must not allow the vector $\mathbf{a}$ to have a component in the direction (or against it) of $\mathbf{n}$. Therefore, $\mathbf{a}=\mathbf{a}_\perp$, which just means that $\mathbf{a}_\parallel=0$. So, (36) becomes

$$R^\dagger\mathbf{p}R+\mathbf{a}_\perp = \mathbf{p}\,. \qquad (37)$$

Let's now perform our standard decomposition of $\mathbf{p}$,

$$\mathbf{p} = \mathbf{p}_\parallel+\mathbf{p}_\perp\,. \qquad (38)$$

Therefore,

$$R^\dagger(\mathbf{p}_\| + \mathbf{p}_\perp)R + \mathbf{a}_\perp = \mathbf{p}\,, \tag{39}$$

which becomes

$$\mathbf{p}_\| + R^\dagger \mathbf{p}_\perp R + \mathbf{a}_\perp = \mathbf{p}\,. \tag{40}$$

Or

$$\mathbf{p}_\perp R^2 + \mathbf{a}_\perp = \mathbf{p}_\perp\,, \tag{41}$$

which simplifies to

$$\mathbf{p}_\perp = \mathbf{a}_\perp(1 - R^2)^{-1}\,. \tag{42}$$

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The rest of this demonstration is little more than just complex numbers and trigonometry. According to the text, we assume that R has the form

$$R = e^{\frac{1}{2} i\mathbf{n}\phi}\,. \tag{43}$$

Then

$$R^2 = e^{i\mathbf{n}\phi}\,. \tag{44}$$

Thus

$$\begin{aligned}
\mathbf{b}_\perp &= \mathbf{a}_\perp(1 - e^{i\mathbf{n}\phi})^{-1} \\
&= \mathbf{a}_\perp \frac{(1 - e^{-i\mathbf{n}\phi})}{(1 - e^{i\mathbf{n}\phi})(1 - e^{-i\mathbf{n}\phi})} \\
&= \mathbf{a}_\perp \frac{1 - e^{-i\mathbf{n}\phi}}{1 - (e^{-i\mathbf{n}\phi} + e^{i\mathbf{n}\phi}) + 1} \\
&= \tfrac{1}{2}\mathbf{a}_\perp \frac{1 - e^{-i\mathbf{n}\phi}}{1 - \cos\phi} \\
&= \tfrac{1}{2}\mathbf{a}_\perp \frac{1 - (\cos\phi - i\mathbf{n}\sin\phi)}{1 - \cos\phi} \\
&= \tfrac{1}{2}\mathbf{a}_\perp \left[1 + i\mathbf{n}\frac{\sin\phi}{1 - \cos\phi}\right] \\
&= \tfrac{1}{2}\mathbf{a}_\perp \left[1 + i\mathbf{n}\cot\tfrac{1}{2}\phi\right] \\
&= \tfrac{1}{2}\left[\mathbf{a}_\perp + \mathbf{n}\times\mathbf{a}\cot\tfrac{1}{2}\phi\right]\,.
\end{aligned} \tag{45}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.