Problems from Chapter 5, Section 5

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1 Rigid Motions and Frames of Reference

Problem (5.1) page 315–316 of NFCM [1].

For a unitary rotor $R = R(t)$ with the parameterizations

$$R = e^{(1/2)i\mathbf{a}} = \alpha + i\boldsymbol{\beta} = \alpha(1 + i\boldsymbol{\gamma})\,, \tag{1}$$

show that the rotational velocity $i\boldsymbol{\omega} = 2R^\dagger \dot{R}$ has the following parametric expressions:

$$\boldsymbol{\omega} = 2(\alpha\boldsymbol{\beta} - \dot{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})\,, \tag{2}$$

where $\alpha^2 + \boldsymbol{\beta}^2 = 1$ and $\alpha\dot{\alpha} = -\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}$. Then

$$\boldsymbol{\omega} = 2\left(\frac{\dot{\boldsymbol{\gamma}} + \boldsymbol{\gamma} \times \dot{\boldsymbol{\gamma}}}{1 + \gamma^2}\right)\,, \tag{3}$$

and, lastly,

$$\boldsymbol{\omega} = \mathbf{n}\dot{a} + \dot{\mathbf{n}} \sin a + \mathbf{n} \times \dot{\mathbf{n}}(1 - \cos a)\,, \tag{4}$$

where $\mathbf{n}^2 = 1$, $\mathbf{a} = a\mathbf{n}$, and $\mathbf{n} \times \dot{\mathbf{n}} = i\dot{\mathbf{n}}\mathbf{n}$.

First, we establish (2):

$$\begin{aligned}
\boldsymbol{\omega} &= -2iR^\dagger \dot{R} \\
&= -2i(\alpha - i\boldsymbol{\beta})(\dot{\alpha} + i\boldsymbol{\beta}) \\
&= -2i\langle\, (\alpha - i\boldsymbol{\beta})(\dot{\alpha} + i\boldsymbol{\beta})\,\rangle_2 \\
&= -2i\langle\, \alpha\dot{\alpha} + i\alpha\dot{\boldsymbol{\beta}} - i\dot{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta}\dot{\boldsymbol{\beta}}\,\rangle_2 \\
&= 2(\alpha\dot{\boldsymbol{\beta}} - \dot{\alpha}\boldsymbol{\beta} - i\boldsymbol{\beta} \wedge \dot{\boldsymbol{\beta}}) \\
&= 2(\alpha\dot{\boldsymbol{\beta}} - \dot{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})\,.
\end{aligned} \tag{5}$$

To establish (3), substitute into (2) $\boldsymbol{\beta} = \alpha\boldsymbol{\gamma}$, etc, and that $\alpha^2 = 1/(1 + \gamma^2)$.

To establish (4), make the substitutions

$$\alpha = \cos \tfrac{1}{2}a\,, \qquad \boldsymbol{\beta} = \mathbf{n} \sin \tfrac{1}{2}a\,, \tag{6a}$$

$$\dot{\alpha} = -\tfrac{1}{2}\dot{a} \sin \tfrac{1}{2}a\,, \quad \dot{\boldsymbol{\beta}} = \dot{\mathbf{n}} \sin \tfrac{1}{2}a + \mathbf{n}\tfrac{1}{2}\dot{a} \cos \tfrac{1}{2}a\,, \tag{6b}$$

into (2) , to get

$$\boldsymbol{\omega} = 2[\cos\tfrac{1}{2}a(\dot{\mathbf{n}}\sin\tfrac{1}{2}a + \mathbf{n}\tfrac{1}{2}\dot{a}\cos\tfrac{1}{2}a) - (-\tfrac{1}{2}\dot{a}\sin\tfrac{1}{2}a)\,\mathbf{n}\sin\tfrac{1}{2}a \\ + (\mathbf{n}\sin\tfrac{1}{2}a)\times(\dot{\mathbf{n}}\sin\tfrac{1}{2}a + \mathbf{n}\tfrac{1}{2}\dot{a}\cos\tfrac{1}{2}a)\,]\,, \tag{7}$$

which, on employing the trigonometric identities

$$\sin^2\tfrac{1}{2}a + \cos^2\tfrac{1}{2}a = 1\,, \\ \sin^2\tfrac{1}{2}a = \tfrac{1}{2}(1-\cos a)\,, \tag{8}$$

becomes

$$\boldsymbol{\omega} = \mathbf{n}\dot{a} + \dot{\mathbf{n}}\sin a + \mathbf{n}\times\dot{\mathbf{n}}\,(1-\cos a)\,. \tag{9}$$

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Problem (5.2) page 315:

Four time-dependent unitary spinors R_k satisfying $\dot{R}_k = \frac{1}{2}R_k\mathbf{\Omega}_k$ are related by the equation

$$R_4 = R_1R_2R_3\,. \tag{10}$$

Show that their rotational velocities are related by the equation

$$\mathbf{\Omega}_4 = R_3^\dagger(R_2^\dagger\mathbf{\Omega}_1R_2 + \mathbf{\Omega}_2)R_3 + \mathbf{\Omega}_3\,. \tag{11}$$

Proof:

First,

$$\mathbf{\Omega}_4 = 2R_4^\dagger\dot{R}_4\,. \tag{12}$$

On differentiating (10), we get

$$\dot{R}_4 = \dot{R}_1R_2R_3 + R_1\dot{R}_2R_3 + R_1R_2\dot{R}_3 \\ = \tfrac{1}{2}[\,R_1\mathbf{\Omega}_1R_2R_3 + R_1R_2\mathbf{\Omega}_2R_3 + R_1R_2R_3\mathbf{\Omega}_3\,]\,. \tag{13}$$

Therefore,

$$\mathbf{\Omega}_4 = R_3^\dagger R_2^\dagger R_1^\dagger[\,R_1\mathbf{\Omega}_1R_2R_3 + R_1R_2\mathbf{\Omega}_2R_3 + R_1R_2R_3\mathbf{\Omega}_3\,] \\ = R_3^\dagger(R_2^\dagger\mathbf{\Omega}_1R_2 + \mathbf{\Omega}_2)R_3 + \mathbf{\Omega}_3\,. \tag{14}$$

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Problem (5.3) page 315–316:

For unitary spinor $R = R(t)$ with Eulerian parameterization

$$R = e^{\frac{1}{2}i\boldsymbol{\sigma}_3\psi}e^{\frac{1}{2}i\boldsymbol{\sigma}_1\theta}e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\,, \tag{15}$$

show that the rotational velocity $i\boldsymbol{\omega} = 2R^\dagger \dot{R}$ has the parametric form

$$\boldsymbol{\omega} = \dot{\phi}\boldsymbol{\sigma}_3 + \dot{\theta}\widehat{\boldsymbol{\sigma}_3 \times \mathbf{n}} + \dot{\psi}\mathbf{n}\,, \tag{16}$$

where

$$\mathbf{n} = R^\dagger \boldsymbol{\sigma}_3 R = \boldsymbol{\sigma}_3 \cos\theta - \boldsymbol{\sigma}_2 e^{i\boldsymbol{\sigma}_3\phi}\sin\theta\,, \tag{17}$$

and

$$\widehat{\boldsymbol{\sigma}_3 \times \mathbf{n}} = \frac{\boldsymbol{\sigma}_3 \times \mathbf{n}}{|\,\boldsymbol{\sigma}_3 \times \mathbf{n}\,|} = \boldsymbol{\sigma}_1 e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} = \boldsymbol{\sigma}_1\cos\phi + \boldsymbol{\sigma}_2\sin\phi\,, \tag{18}$$

Proof:

We begin with (11)

$$\boldsymbol{\Omega} = i\boldsymbol{\omega} = R_3^\dagger (R_2^\dagger \boldsymbol{\Omega}_1 R_2 + \boldsymbol{\Omega}_2) R_3 + \boldsymbol{\Omega}_3\,, \tag{19}$$

where

$$R_1 = e^{\frac{1}{2}i\boldsymbol{\sigma}_3\psi}\,, \quad R_2 = e^{\frac{1}{2}i\boldsymbol{\sigma}_1\theta}\,, \quad R_3 = e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\,. \tag{20}$$

So, (19) becomes

$$i\boldsymbol{\omega} = e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}(e^{-\frac{1}{2}i\boldsymbol{\sigma}_1\theta} i\boldsymbol{\omega}_1 e^{\frac{1}{2}i\boldsymbol{\sigma}_1\theta} + i\boldsymbol{\omega}_2) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + i\boldsymbol{\omega}_3\,. \tag{21}$$

We can drop the common factor of i,

$$\boldsymbol{\omega} = e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}(e^{-\frac{1}{2}i\boldsymbol{\sigma}_1\theta} \boldsymbol{\omega}_1 e^{\frac{1}{2}i\boldsymbol{\sigma}_1\theta} + \boldsymbol{\omega}_2) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + \boldsymbol{\omega}_3\,. \tag{22}$$

Next, we use the fact that

$$\boldsymbol{\omega}_1 = \boldsymbol{\sigma}_3\dot{\psi}\,, \quad \boldsymbol{\omega}_2 = \boldsymbol{\sigma}_1\dot{\theta}\,, \quad \boldsymbol{\omega}_3 = \boldsymbol{\sigma}_3\dot{\phi}\,, \tag{23}$$

to get

$$\boldsymbol{\omega} = e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\big(e^{-\frac{1}{2}i\boldsymbol{\sigma}_1\theta} \boldsymbol{\sigma}_3\dot{\psi} e^{\frac{1}{2}i\boldsymbol{\sigma}_1\theta} + \boldsymbol{\sigma}_1\dot{\theta}\big) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + \boldsymbol{\sigma}_3\dot{\phi}\,. \tag{24}$$

Pulling $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$ to the left, we get

$$\begin{aligned}
\boldsymbol{\omega} &= \boldsymbol{\sigma}_3\dot{\psi} e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\big(e^{i\boldsymbol{\sigma}_1\theta}\big) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\big(\boldsymbol{\sigma}_1\dot{\theta}\big) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + \boldsymbol{\sigma}_3\dot{\phi} \\
&= \dot{\phi}\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3\dot{\psi} e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\big(e^{i\boldsymbol{\sigma}_1\theta}\big) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + \big(\boldsymbol{\sigma}_1\dot{\theta}\big) e^{i\boldsymbol{\sigma}_3\phi} \\
&= \dot{\phi}\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3\dot{\psi} e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}(\cos\theta + i\boldsymbol{\sigma}_1\sin\theta) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi} + \big(\boldsymbol{\sigma}_1\dot{\theta}\big)(\cos\phi + i\boldsymbol{\sigma}_3\sin\phi)\,.
\end{aligned}$$

Continuing, we have that

$$\begin{aligned}
\boldsymbol{\omega} &= \dot{\phi}\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3\dot{\psi}[\,\cos\theta + e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\phi}(i\boldsymbol{\sigma}_1\sin\theta) e^{\frac{1}{2}i\boldsymbol{\sigma}_3\phi}\,] + \big(\boldsymbol{\sigma}_1\dot{\theta}\big)(\cos\phi + i\boldsymbol{\sigma}_3\sin\phi) \\
&= \dot{\phi}\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3\dot{\psi}[\,\cos\theta + (i\boldsymbol{\sigma}_1\sin\theta) e^{i\boldsymbol{\sigma}_3\phi}\,] + \dot{\theta}\big(\widehat{\boldsymbol{\sigma}_3 \times \mathbf{n}}\big) \\
&= \dot{\phi}\boldsymbol{\sigma}_3 + \dot{\psi}[\,\boldsymbol{\sigma}_3\cos\theta - \boldsymbol{\sigma}_2 e^{i\boldsymbol{\sigma}_3\phi}\sin\theta\,] + \dot{\theta}\big(\widehat{\boldsymbol{\sigma}_3 \times \mathbf{n}}\big) \\
&= \dot{\phi}\boldsymbol{\sigma}_3 + \dot{\psi}\mathbf{n} + \dot{\theta}\widehat{\boldsymbol{\sigma}_3 \times \mathbf{n}}\,.
\end{aligned}$$

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Problem (5.5) page 316:

FIll in the steps in the derivation of Eq. (5.17) in the text.

Proof of the relation (see p. 310, Eq. (5.17)).

$$\mathcal{R}(\boldsymbol{\omega} \times \mathbf{x}) = \mathcal{R}(\boldsymbol{\omega}) \times \mathcal{R}(\mathbf{x})\,, \tag{25}$$

under the rotation

$$\mathcal{R}(\mathbf{x}) = R^{\dagger}\mathbf{x}R\,. \tag{26}$$

Proof: We begin with the identity

$$\mathbf{a} \times \mathbf{b} = -i\mathbf{a} \wedge \mathbf{b} = -i\tfrac{1}{2}(\mathbf{ab} - \mathbf{ba})\,. \tag{27}$$

$$\begin{aligned}
\mathcal{R}(\boldsymbol{\omega} \times \mathbf{x}) &= R^{\dagger}[\,-i\tfrac{1}{2}(\boldsymbol{\omega}\mathbf{x} - \mathbf{x}\boldsymbol{\omega})\,]R \\
&= -i\tfrac{1}{2}[\,R^{\dagger}(\boldsymbol{\omega}\mathbf{x} - \mathbf{x}\boldsymbol{\omega})R\,] \\
&= -i\tfrac{1}{2}[\,R^{\dagger}\boldsymbol{\omega}RR^{\dagger}\mathbf{x}R - R^{\dagger}\mathbf{x}RR^{\dagger}\boldsymbol{\omega}R\,] \\
&= -i\tfrac{1}{2}[\,\mathcal{R}(\boldsymbol{\omega})\mathcal{R}(\mathbf{x}) - \mathcal{R}(\mathbf{x})\mathcal{R}(\boldsymbol{\omega})\,] \\
&= \mathcal{R}(\boldsymbol{\omega}) \times \mathcal{R}(\mathbf{x})\,.
\end{aligned} \tag{28}$$

Now, $\mathcal{R}(\boldsymbol{\omega}) = \boldsymbol{\omega}'$ and $\mathbf{x}' = \mathcal{R}(\mathbf{x}) + \mathbf{a}$, hence

$$\mathcal{R}(\boldsymbol{\omega} \times \mathbf{x}) = \boldsymbol{\omega}' \times (\mathbf{x}' - \mathbf{a})\,. \tag{29}$$

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Problem (5.6) page 316:

Derive Eqs. (5.19) and (5.23) directly, using

$$\dot{R} = \tfrac{1}{2}\mathbf{\Omega}R\,. \tag{30}$$

Proof of Eq. (5.19) (see p. 311). We begin with

$$\mathbf{x}' = \mathcal{R}(\mathbf{x}) + \mathbf{a}\,, \tag{31}$$

and differentiate, to get

$$\begin{aligned}
\dot{\mathbf{x}}' &= \dot{\mathcal{R}}(\mathbf{x}) + \mathcal{R}(\dot{\mathbf{x}}) + \dot{\mathbf{a}} \\
&= \dot{R}^{\dagger}(\mathbf{x})R + R^{\dagger}(\mathbf{x})\dot{R} + \mathcal{R}(\dot{\mathbf{x}}) + \dot{\mathbf{a}} \\
&= (-\tfrac{1}{2}R^{\dagger}\mathbf{\Omega})(\mathbf{x})R + R^{\dagger}(\mathbf{x})(\tfrac{1}{2}\mathbf{\Omega}R) + \mathcal{R}(\dot{\mathbf{x}}) + \dot{\mathbf{a}} \\
&= R^{\dagger}[\tfrac{1}{2}(\mathbf{x}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{x})]R + \mathcal{R}(\dot{\mathbf{x}}) + \dot{\mathbf{a}} \\
&= R^{\dagger}(\mathbf{x} \cdot \mathbf{\Omega})R + \mathcal{R}(\dot{\mathbf{x}}) + \dot{\mathbf{a}} \\
&= \mathcal{R}(\mathbf{x} \cdot \mathbf{\Omega} + \dot{\mathbf{x}}) + \dot{\mathbf{a}}\,.
\end{aligned} \tag{32}$$

Equation (5.23) is left to the reader.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.