Notes on Chapter 5, Section 5

P. Reany

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1 Rigid Motions and Frames of Reference

These notes cover pages 306 to 316 of NFCM [1].

We begin on page 307. Establish Eq. (5.7):

$$\dot{R}^\dagger = \frac{d}{dt}(R^\dagger) = \left(\frac{dR}{dt}\right)^\dagger . \tag{1}$$

First, we take stock of our resources. Eq. (5.5) gives us

$$\dot{R} = \tfrac{1}{2}R\mathbf{\Omega} , \tag{2}$$

where $\mathbf{\Omega}$ is a bivector. Second, we have the unitarity of R, given by Eq. (5.9), gives us

$$R^\dagger R = 1 . \tag{3}$$

On differentiating this last equation by time, we get

$$\frac{d}{dt}(R^\dagger)R + R^\dagger \frac{dR}{dt} = 0 . \tag{4}$$

From this and (2), we get

$$\frac{d}{dt}(R^\dagger)R = -R^\dagger \tfrac{1}{2}R\mathbf{\Omega} = -\tfrac{1}{2}\mathbf{\Omega} . \tag{5}$$

Now, multiply on the right by $R^\dagger$, and continue.

$$\frac{d}{dt}(R^\dagger) = -\tfrac{1}{2}\mathbf{\Omega}R^\dagger = \tfrac{1}{2}\mathbf{\Omega}^\dagger R^\dagger = (\tfrac{1}{2}R\mathbf{\Omega})^\dagger = (\dot{R})^\dagger . \tag{6}$$

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We have the expression for $\mathbf{r}$ in Eq. (5.4):

$$\mathbf{r} = R^\dagger \mathbf{r}_0 R . \tag{7}$$

Differentiating this by time gives us

$$\dot{\mathbf{r}} = \dot{R}^\dagger \mathbf{r}_0 R + R^\dagger \mathbf{r}_0 \dot{R} . \tag{8}$$

We will reduce this last equation in two steps. First, we will use (2) to get rid of $\dot{R}$, and then use (7) to get rid of $\mathbf{r}_0$:

$$\begin{aligned}
\dot{\mathbf{r}} &= \dot{R}^\dagger \mathbf{r}_0 R + R^\dagger \mathbf{r}_0 \dot{R} \\
&= -\tfrac{1}{2}\boldsymbol{\Omega} R^\dagger \mathbf{r}_0 R + R^\dagger \mathbf{r}_0 \tfrac{1}{2} R \boldsymbol{\Omega} \\
&= -\tfrac{1}{2}\boldsymbol{\Omega}\mathbf{r} + \tfrac{1}{2}\mathbf{r}\boldsymbol{\Omega} \\
&= \tfrac{1}{2}(\mathbf{r}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{r}) \\
&= \mathbf{r} \cdot \boldsymbol{\Omega} \\
&= \boldsymbol{\omega} \times \mathbf{r}\,. \qquad (9)
\end{aligned}$$

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Proof of the relation (see p. 310, Eq. (5.17)).

$$\mathcal{R}(\boldsymbol{\omega} \times \mathbf{x}) = \mathcal{R}(\boldsymbol{\omega}) \times \mathcal{R}(\mathbf{x})\,, \qquad (10)$$

under the rotation

$$\mathcal{R}(\mathbf{x}) = R^\dagger \mathbf{x} R\,. \qquad (11)$$

Proof: We begin with the identity

$$\mathbf{a} \times \mathbf{b} = -i\mathbf{a} \wedge \mathbf{b} = -i\tfrac{1}{2}(\mathbf{ab} - \mathbf{ba})\,. \qquad (12)$$

$$\begin{aligned}
\mathcal{R}(\boldsymbol{\omega} \times \mathbf{x}) &= R^\dagger [\,-i\tfrac{1}{2}(\boldsymbol{\omega}\mathbf{x} - \mathbf{x}\boldsymbol{\omega})\,] R \\
&= -i\tfrac{1}{2}[\,R^\dagger(\boldsymbol{\omega}\mathbf{x} - \mathbf{x}\boldsymbol{\omega})R\,] \\
&= -i\tfrac{1}{2}[\,R^\dagger\boldsymbol{\omega}RR^\dagger\mathbf{x}R - R^\dagger\mathbf{x}RR^\dagger\boldsymbol{\omega}R\,] \\
&= -i\tfrac{1}{2}[\,\mathcal{R}(\boldsymbol{\omega})\mathcal{R}(\mathbf{x}) - \mathcal{R}(\mathbf{x})\mathcal{R}(\boldsymbol{\omega})\,] \\
&= \mathcal{R}(\boldsymbol{\omega}) \times \mathcal{R}(\mathbf{x})\,. \qquad (13)
\end{aligned}$$

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References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.