Notes on Chapter 9 Section 1

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1 Problem

These notes cover pages 574 to 588 of NFCM [1].

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Show that

$$(MN)^\sim = \widetilde{N}\widetilde{M}\,, \tag{1}$$

given that

$$M^\sim = \langle\, M^\dagger \,\rangle_+ - \langle\, M^\dagger \,\rangle_- = (\langle\, M \,\rangle_+ - \langle\, M \,\rangle_-)^\dagger\,. \tag{2}$$

My plan is to use (2) to separately expand both sides of (1) and then meet in the middle.

Every multivector A can be expanded as the sum of its even- and odd-graded parts,

$$A = A_+ + A_-\,. \tag{3}$$

Thus

$$\begin{aligned}\widetilde{N}\widetilde{M} &= (N_+ + N_-)^\sim (M_+ + M_-)^\sim \\ &= (N_+^\dagger - N_-^\dagger)(M_+^\dagger - M_-^\dagger) \\ &= N_+^\dagger M_+^\dagger - N_+^\dagger M_-^\dagger - N_-^\dagger M_+^\dagger + N_-^\dagger M_-^\dagger \\ &= N_+^\dagger M_+^\dagger + + N_-^\dagger M_-^\dagger - (N_+^\dagger M_-^\dagger + N_-^\dagger M_+^\dagger)\,.\end{aligned} \tag{4}$$

And

$$\begin{aligned}MN &= (M_+ + M_-)(N_+ + N_-) \\ &= M_+N_+ + M_+N_- + M_-N_+ + M_-N_- \\ &= (M_+N_+ + M_-N_-) + (M_+N_- + M_-N_+)\,,\end{aligned} \tag{5}$$

so,

$$\begin{aligned}(MN)^\sim &= (M_+N_+ + M_-N_-)^\dagger - (M_+N_- + M_-N_+)^\dagger \\ &= N_+^\dagger M_+^\dagger + + N_-^\dagger M_-^\dagger - (N_+^\dagger M_-^\dagger + N_-^\dagger M_+^\dagger)\,.\end{aligned} \tag{6}$$

And so we have established (1).

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Here I will show how to find (1.21)

$$\pm L = \frac{1+\gamma+\gamma\mathbf{v}/c}{[2(1+\gamma)]^{1/2}}\,, \tag{7}$$

from (1.20a)

$$L^2 = \gamma\Big(1+\frac{\mathbf{v}}{c}\Big)\,, \tag{8}$$

by taking the square root. I can do this by employing the methods of the unipodal algebra. But, first, let's simplify the expression in (8). We rewrite it as

$$L^2 = a + b\hat{\mathbf{v}}\,, \tag{9}$$

where $a=\gamma$, $b=\gamma v/c$, $\hat{\mathbf{v}}$ is a unit vector, and $a^2-b^2=1$.

Lemma 1:

Here we deal with the typical relations common to special relativity regarding γ and v/c.

Let

$$Z_+ = (a+b)^{\frac{1}{2}} + (a-b)^{\frac{1}{2}}\,. \tag{10}$$

On squaring this and simplifying it, we get

$$Z_+^2 = 2(\gamma+1)\,. \tag{11}$$

On taking the square root of this we get

$$Z_+ = [2(\gamma+1)]^{\frac{1}{2}}\,. \tag{12}$$

By similar reasoning, if we set

$$Z_- = (a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}}\,. \tag{13}$$

On squaring this and simplifying it, we get

$$Z_-^2 = 2(\gamma-1)\,. \tag{14}$$

On taking the square root of this we get

$$Z_- = [2(\gamma-1)]^{\frac{1}{2}}\,. \tag{15}$$

Now, L^2 in (9) is just a unipodal number in standard basis $\{1,\hat{\mathbf{v}}\}$, and we can directly take its square root once we've put it in the idempotent basis

$$u_+ = \tfrac{1}{2}(1+\hat{\mathbf{v}}) \qquad \text{and} \qquad u_- = \tfrac{1}{2}(1-\hat{\mathbf{v}})\,. \tag{16}$$

It's easy to show that

$$u_+ + u_- = 1 \qquad \text{and} \qquad u_+ - u_- = \hat{\mathbf{v}}\,. \tag{17}$$

Now, L^2 in (9) can be put into the form

$$L^2 = a(u_+ + u_-) + b(u_+ - u_-)\,, \tag{18}$$

which simplifies to

$$L^2 = (a+b)u_+ + (a-b)u_- \,, \tag{19}$$

Now, we just take the square root:

$$\pm L = (a+b)^{\frac{1}{2}}u_+ + (a-b)^{\frac{1}{2}}u_- \,. \tag{20}$$

Great, but now we've got to return to standard basis:

$$\begin{aligned}\pm L &= (a+b)^{\frac{1}{2}}\tfrac{1}{2}(1+\hat{\mathbf{v}}) + (a-b)^{\frac{1}{2}}\tfrac{1}{2}(1-\hat{\mathbf{v}}) \\ &= \tfrac{1}{2}((a+b)^{\frac{1}{2}} + (a-b)^{\frac{1}{2}}) + \tfrac{1}{2}((a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}})\hat{\mathbf{v}} \\ &= \left(\frac{a+1}{2}\right)^{\frac{1}{2}} + \left(\frac{a-1}{2}\right)^{\frac{1}{2}}\hat{\mathbf{v}} \,. \end{aligned} \tag{21}$$

Now we use the results of the lemma above:

$$\pm L = \tfrac{1}{2}[2(\gamma+1)]^{\frac{1}{2}} + \tfrac{1}{2}[2(\gamma-1)]^{\frac{1}{2}}\hat{\mathbf{v}} = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}} + \left(\frac{\gamma-1}{2}\right)^{\frac{1}{2}}\hat{\mathbf{v}} \,. \tag{22}$$

On multiplying the numerator and denominator of the RHS side by $[2(\gamma+1)]^{\frac{1}{2}}$, and simplifying, using that $\gamma^2 - 1 = \gamma v/c$, we get

$$\pm L = \frac{1 + \gamma + \gamma\mathbf{v}/c}{[2(1+\gamma)]^{1/2}} \,. \tag{23}$$

Lemma 2:

The Fundamental Theorem of Exponentiation in the unipodal algebra:

$$e^{x_+u_+ + x_-u_-} = e^{x_+}u_+ + e^{x_-}u_- \,. \tag{24}$$

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My goal is to show that the text equation (1.29) is true:

$$\tanh\, a = \frac{v}{c} \,. \tag{25}$$

We begin with

$$L^2 = \gamma + \gamma v\hat{\mathbf{v}}/c \,. \tag{26}$$

On setting $L^2 = e^{\mathbf{a}}$, we need a way to expand $e^{\mathbf{a}}$.

We start as we did last time. With $\mathbf{a} = a\hat{\mathbf{a}}$:

$$u_+ = \tfrac{1}{2}(1+\hat{\mathbf{a}}) \qquad \text{and} \qquad u_- = \tfrac{1}{2}(1-\hat{\mathbf{a}}) \,. \tag{27}$$

It's easy to show that

$$u_+ + u_- = 1 \qquad \text{and} \qquad u_+ - u_- = \hat{\mathbf{a}} \,. \tag{28}$$

To apply (24), we set

$$\mathbf{a} = a\hat{\mathbf{a}} = a(u_+ - u_-) = au_+ - au_- \,. \tag{29}$$

Then,

$$e^{\mathbf{a}} = e^{au_+ + (-a)u_-} = e^{a}u_+ + e^{-a}u_- \,. \tag{30}$$

Expresssing this result in the standard basis, we get

$$e^{\mathbf{a}} = \tfrac{1}{2}(e^a + e^{-a}) + \tfrac{1}{2}(e^a - e^{-a})\hat{\mathbf{a}} = \cosh a + \hat{\mathbf{a}} \sinh a\,. \tag{31}$$

Hence, from the components of (26),

$$\tanh\, a = \frac{\sinh a}{\cosh a} = \frac{\gamma v/c}{\gamma} = \frac{v}{c}\,. \tag{32}$$

And from (26) and (31), we get that

$$\cosh a = \gamma\,. \tag{33}$$

Lemma 3:

This lemma is a corollary of Lemma 2. Because the scalar multiplier in the exponent of the fundamental theorem of exponentiation in the unipodal algebra can be complex, the theorem can be used to prove that

$$e^{-\frac{1}{2}i\mathbf{b}} = \cos \tfrac{1}{2}b - i\hat{\mathbf{b}} \sin b\,. \tag{34}$$

Useful intermediate results are that

$$\cosh\,(-\tfrac{1}{2}ib\hat{\mathbf{b}}) = \cos \tfrac{1}{2}b \qquad \text{and} \qquad \sinh\,(-\tfrac{1}{2}ib\hat{\mathbf{b}}) = -i\hat{\mathbf{b}} \sin \tfrac{1}{2}b = -i \sin \tfrac{1}{2}\mathbf{b}\,. \tag{35}$$

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Here we will show that a general Lorentz transformation L can be factored as

$$L = RB\,, \tag{36}$$

where R and B satisfy the properties:

$$\widetilde{R} = R^\dagger\,, \tag{37}$$

and

$$B^\dagger = B\,. \tag{38}$$

We begin by defining the multivector A:

$$A = LL^\dagger\,. \tag{39}$$

From (1.18b), we have that

$$L\widetilde{L} = \widetilde{L}L = 1\,. \tag{40}$$

Then

$$\begin{aligned} A\widetilde{A} &= LL^\dagger(LL^\dagger)^\sim \\ &= LL^\dagger\widetilde{L}^\dagger\widetilde{L} \\ &= L(\widetilde{L}L)^\dagger\widetilde{L} \\ &= 1\,. \end{aligned} \tag{41}$$

And now,

$$A^\dagger = (LL^\dagger)^\dagger = L^{\dagger\dagger}L^\dagger = LL^\dagger = A\,. \tag{42}$$

From the material at the bottom of page 581, we have determined that A is a particular kind of Lorentz transformation, called a *Boost*, which has the form of a scalar plus a vector. We found that taking the square root of (9) produced (22).

We've been asked to prove some properties of B, which is the square root of A. Thus, if we start with A in the form

$$A = a + b\hat{\mathbf{v}}\,, \tag{43}$$

and since $A\widetilde{A} = 1$, then $a^2 - b^2 = 1$. On taking the square root of A, we get that

$$B \equiv A^{\frac{1}{2}} = \left(\frac{a+1}{2}\right)^{\frac{1}{2}} + \left(\frac{a-1}{2}\right)^{\frac{1}{2}}\hat{\mathbf{v}}\,. \tag{44}$$

Therefore,

$$\begin{aligned} B\widetilde{B} &= \left[\left(\frac{a+1}{2}\right)^{\frac{1}{2}} + \left(\frac{a-1}{2}\right)^{\frac{1}{2}}\hat{\mathbf{v}}\right]\left[\left(\frac{a+1}{2}\right)^{\frac{1}{2}} - \left(\frac{a-1}{2}\right)^{\frac{1}{2}}\hat{\mathbf{v}}\right] \\ &= \frac{a+1}{2} - \frac{a-1}{2} \\ &= 1\,. \end{aligned} \tag{45}$$

Now, since B is a scalar plus a vector, it's invariant under the reversion operation, hence,

$$B^\dagger = B\,. \tag{46}$$

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Our job now is to show that we can solve for $\mathbf{b}$ in

$$B_2 B_1 = B_3 e^{-\frac{1}{2}i\mathbf{b}}\,, \tag{47}$$

where

$$B_k = a_k + b_k\mathbf{v}_k = \frac{1+\gamma_k}{[2(1+\gamma_k)]^{1/2}} + \frac{\gamma_k/c}{[2(1+\gamma_k)]^{1/2}}\mathbf{v}_k\,. \tag{48}$$

to get

$$\tan\tfrac{1}{2}\mathbf{b} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{c^2(1+\gamma_1^{-1})(1+\gamma_2^{-1}) + \mathbf{v}_1\cdot\mathbf{v}_2}\,. \tag{49}$$

We will also need (34) and

$$\mathbf{v}_3 = \frac{\mathbf{v}_1 + \mathbf{v}_2 + (\gamma_2^{-1} - 1)\hat{\mathbf{v}}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{v}}_2)}{1 + \mathbf{v}_1\cdot\mathbf{v}_2/c^2}\,. \tag{50}$$

Now, we plug in, expand, and then take the appropriate graded parts as needed.

$$(a_2 + b_2\mathbf{v}_2)(a_1 + b_1\mathbf{v}_1) = (a_3 + b_3\mathbf{v}_3)(\cos\tfrac{1}{2}\mathbf{b} - i\sin\tfrac{1}{2}\mathbf{b})\,, \tag{51}$$

Expanding, we get

$$a_2a_1 + a_2b_1\mathbf{v}_1 + b_2a_1\mathbf{v}_2 + b_2b_1\mathbf{v}_2\mathbf{v}_1 = a_3\cos\tfrac{1}{2}\mathbf{b} - ia_3\sin\tfrac{1}{2}\mathbf{b} + b_3(\cos\tfrac{1}{2}\mathbf{b})\mathbf{v}_3 - ib_3\mathbf{v}_3\sin\tfrac{1}{2}\mathbf{b}\,. \tag{52}$$

Now, we equate the scalar parts:

$$a_2a_1 + b_2b_1\mathbf{v}_2\cdot\mathbf{v}_1 = a_3\cos\tfrac{1}{2}\mathbf{b}\,, \tag{53}$$

from which we get

$$\cos \tfrac{1}{2}\mathbf{b} = \frac{a_2 a_1 + b_2 b_1 \mathbf{v}_2 \cdot \mathbf{v}_1}{a_3} \,. \tag{54}$$

Now, we equate the bivector parts:

$$b_2 b_1 \mathbf{v}_2 \wedge \mathbf{v}_1 = -i a_3 \sin \tfrac{1}{2}\mathbf{b} \,, \tag{55}$$

from which we get

$$\sin \tfrac{1}{2}\mathbf{b} = \frac{b_2 b_1 \mathbf{v}_1 \times \mathbf{v}_2}{a_3} \,. \tag{56}$$

From there we can form the tangent of $\frac{1}{2}\mathbf{b}$:

$$\tan \tfrac{1}{2}\mathbf{b} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\dfrac{a_1 a_2}{b_1 b_2} + \mathbf{v}_1 \cdot \mathbf{v}_2} \,. \tag{57}$$

Given that

$$\frac{a_1}{b_1} = \frac{1 + \gamma_1}{\gamma_1 / c} \,, \tag{58}$$

and that

$$\frac{a_2}{b_2} = \frac{1 + \gamma_2}{\gamma_2 / c} \,, \tag{59}$$

then

$$\frac{a_1 a_2}{b_1 b_2} = c^2 (1 + \gamma_1^{-1})(1 + \gamma_2^{-1}) \,. \tag{60}$$

Plugging this result into (57) yields (49).

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.