Problem 1.1 on Page 588

P. Reany

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1 Introduction

On page 588 of NFCM [1], we find problem (1.1): Given that $L^\dagger = L$, $L\tilde{L} = 1$, and

$$L^2 = \gamma(1 + \mathbf{v}/c)\,, \tag{1}$$

show that

$$\pm L = \frac{1+\gamma+\gamma\mathbf{v}/c}{[2(\gamma+1)]^{1/2}} = \left(\frac{\gamma+1}{2}\right)^{1/2} + \;\hat{\mathbf{v}}\left(\frac{\gamma-1}{2}\right)^{1/2}\,. \tag{2}$$

2 Solution

Ansatz: Let's try L is the form

$$L = \alpha + \beta\mathbf{v}/c\,, \tag{3}$$

where α and β are to be determined by the given information. Now,

$$\tilde{L} = \alpha - \beta\mathbf{v}/c\,. \tag{4}$$

Therefore,

$$L\tilde{L} = \alpha^2 - \beta^2 v^2/c^2\,, \tag{5}$$

and

$$L^2 = (\alpha^2 + \beta^2 v^2/c^2) + 2\alpha\beta\mathbf{v}/c = \gamma(1+\mathbf{v}/c)\,, \tag{6}$$

where we used (1). From this last equation we get

$$\alpha^2 - \beta^2 v^2/c^2 = 1\,, \tag{7a}$$
$$\alpha^2 + \beta^2 v^2/c^2 = \gamma\,. \tag{7b}$$

From these we get

$$\alpha^2 = \frac{\gamma+1}{2} \tag{8a}$$
$$\beta^2 = \frac{\gamma-1}{2v^2/c^2}\,. \tag{8b}$$

Using these last values into (3), we have

$$\pm L = \left(\frac{\gamma+1}{2}\right)^{1/2} + \;\hat{\mathbf{v}}\left(\frac{\gamma-1}{2}\right)^{1/2}\,. \tag{9}$$

I need to show why we can't get L in the form of, say,

$$L = \left(\frac{\gamma+1}{2}\right)^{1/2} - \hat{\mathbf{v}}\left(\frac{\gamma-1}{2}\right)^{1/2}, \tag{10}$$

that is, of the terms having opposite signs. On squaring (10) and using that

$$\sqrt{\gamma^2 - 1} = \frac{v\gamma}{c}, \tag{11}$$

we get

$$L^2 = \gamma(1 - \mathbf{v}/c), \tag{12}$$

which is not consistent with (1).

To express L in the form of a fraction with denominator $[2(\gamma+1)]^{1/2}$, we are benefitted to know that

$$\gamma^2 - 1 = \frac{v^2\gamma^2}{c^2}. \tag{13}$$

Therefore, we get that

$$\pm L = \frac{1 + \gamma + \gamma\mathbf{v}/c}{[2(\gamma+1)]^{1/2}}. \tag{14}$$

And we are done.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.