Problem 1.2 on Page 588-9

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1 Problem

On page 588-9 of NFCM [1], we find problem (1.2): The Lorentz transformation of velocity.

2 Setup

Lorentz transformation of velocity.

1) Note that Eq. (1.7) applies to the motion of any particle for a sufficiently short time interval Δt. Use the linearity of the Lorentz transformatino (1.4) to prove that

$$\Delta X = \mathcal{L}(\Delta X')\,. \tag{1}$$

Use these results to show that a particle with velocity $\mathbf{v}'$ with respect to the primed frame has velocity $\mathbf{v}$ with respect to the unprimed frame given by

$$\gamma(c+\mathbf{v}) = \mathcal{L}(c+\mathbf{v}') \qquad \text{where} \qquad \gamma = \frac{dt}{dt'}\,. \tag{2}$$

Using the canonical form $\mathcal{L}(X) = LX'L^\dagger$, derive the general result

$$\gamma(c+\mathbf{u}) = cLL^\dagger \qquad \text{where} \qquad \gamma = (1-\mathbf{u}^2/c^2)^{-1/2}\,, \tag{3}$$

and $\mathbf{u}$ is the velocity of the primed frame with respect to the unprimed frame.

3 Proofs

Proof to Eq. (1). Let

$$X_1 = \mathcal{L}(X_1') \qquad \text{and} \qquad X_2 = \mathcal{L}(X_2')\,, \tag{4}$$

then

$$\Delta X = X_2 - X_1 = \mathcal{L}(X_2') - \mathcal{L}(X_1') = \mathcal{L}(X_2' - X_1') = \mathcal{L}(\Delta X')\,. \tag{5}$$

Now, we can take the limit as our Δ's get vanishingly small, to get

$$dX = \mathcal{L}(dX')\,. \tag{6}$$

Now we multiply through by $\dfrac{\gamma}{dt} = \dfrac{1}{dt'}$, to get

$$\gamma\frac{dX}{dt} = \mathcal{L}\big(\frac{dX'}{dt}\big)\,. \tag{7}$$

But

$$X = ct + \mathbf{x}\,, \tag{8a}$$
$$X' = ct' + \mathbf{x}'\,. \tag{8b}$$

Therefore,

$$\frac{dX}{dt} = c + \mathbf{v}\,, \tag{9a}$$
$$\frac{dX'}{dt} = c + \mathbf{v}'\,. \tag{9b}$$

And since

$$\mathcal{L}\big(\frac{dX'}{dt}\big) = L(c + \mathbf{v}')L^\dagger = cLL^\dagger + L\mathbf{v}'L^\dagger\,. \tag{10}$$

Putting all this into (7), we get

$$\gamma(c + \mathbf{v}) = cLL^\dagger + L\mathbf{v}'L^\dagger \tag{11}$$

Now consider if we set $\mathbf{v}$ equal to the velocity of the primed frame as seen by the unprimed frame, which velocity we'll call $\mathbf{u}$. We can imagine this as the velocity of the primed frame's origin. However, in the primed frame, this velocity is, of course, zero, hence $\mathbf{v}' = 0$, thus

$$\gamma(c + \mathbf{u}) = cLL^\dagger\,, \tag{12}$$

where

$$\gamma = (1 - \mathbf{u}^2/c^2)^{-1/2}\,. \tag{13}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.