Problem 1.3 on Page 589

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1 Problem

On page 589 of NFCM [1], we find problem (1.3): Show explicitly that the Lorentz transformations have all the properties of a group.

2 Preliminary Knowledge

First, let's review the properties of a group.

1) We begin with the understanding that a group G is a set of elements, which has a closed binary operator on it. We'll represent this binary operator by $*$, though, often, juxtaposition is enough to represent it. So, if a and b are any two elements of G, then

$$a * b = c \in G\,. \tag{1}$$

2) For any three elements of G, a, b, c, the associative rule holds:

$$a * (b * c) = (a * b) * c\,. \tag{2}$$

The structure that G has so far is deemed worthy of its own name, being referred to as a *semigroup*.

3) If we add to G an identity element for each element, which we'll call e, then for all $a \in G$

$$a * e = e * a = a\,. \tag{3}$$

A semigroup with identity element is called a *monoid*.[1]

4) The last step to get from a monoid to a group is to add-in the inverse property of G, such that for every element a of G, there exists its *inverse* a^{-1}, which has the property:

$$a * a^{-1} = a^{-1} * a = e\,. \tag{4}$$

Now, when doing real mathematics, we usually drop the training wheels of the $*$ and just use juxtaposition, and, when dealing with functions, we use function composition as the binary operation.

[1] To appreciate the difference between a semigroup and a monoid, note that the set of integers under multiplication is a monoid; whereas the set of even integers under multiplication is only a semigroup (it lacks an identity element).

3 Solution

On page 587, we were introduced to the set of Lorentz transformations constituting a group. But we only saw the first step shown – that of the Lorentz transforms as a binary operator on the space of Lorentz transformations, meaning that the composite of two Lorentz transformations is another Lorentz transformation: Let G be the set of Lorentz transformations $\mathcal{L}$. Then,

$$X' = \mathcal{L}_1(X) = L_1 X L_1^\dagger \,, \tag{5}$$

and

$$X'' = \mathcal{L}_2(X') = L_2 X' L_2^\dagger \,. \tag{6}$$

Therefore, by taking the composite transformations

$$X'' = \mathcal{L}_2\mathcal{L}_1(X) = L_2 L_1 X L_1^\dagger L_2^\dagger = (L_2L_1)X(L_2L_1)^\dagger \,. \tag{7}$$

Therefore, there exists $\mathcal{L}_3 \in G$, such that

$$\mathcal{L}_3(\,.\,) = \mathcal{L}_2\mathcal{L}_1(\,.\,) = (L_2L_1)(\,.\,)(L_2L_1)^\dagger \,. \tag{8}$$

1) Thus we have shown that composition of Lorentz transformations of the set G is a closed binary operation on G.

So, now we trudge through the remaining three requirements of a group:

2) For any three elements of G, $\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$, the associative rule holds:

$$(\mathcal{L}_1\mathcal{L}_2)\mathcal{L}_3 = \mathcal{L}_1(\mathcal{L}_2\mathcal{L}_3) \,. \tag{9}$$

This should be intuitively clear to be true, since we have written the Lorentz transformations as a geometric products of a boost with a rotation; therefore the product of Lorentz transformations is associative by construction of the geometric algebra of spacetime operators.

$$(\mathcal{L}_1\mathcal{L}_2)\mathcal{L}_3(X) = (L_1L_2)L_3 X L_3^\dagger (L_1L_2)^\dagger \tag{10}$$

$$= (L_1L_2L_3)X(L_1L_2L_3)^\dagger \,, \tag{11}$$

whereas

$$\mathcal{L}_1(\mathcal{L}_2\mathcal{L}_3)(X) = L_1(L_2L_3)X(L_2L_3)^\dagger L_1^\dagger \tag{12}$$

$$= (L_1L_2L_3)X(L_1L_2L_3)^\dagger \,, \tag{13}$$

Thus we have established (9).

3) The identity element for the Lorentz transformations is obvious the element $\mathcal{L}_e(\,.\,) = 1(\,.\,)1^\dagger$. Thus,

$$\mathcal{L}_e\mathcal{L}(X) = (1L)X(1L)^\dagger = LXL^\dagger = \mathcal{L}(X) \,, \tag{14}$$

and

$$\mathcal{L}\mathcal{L}_e(X) = (L1)X(L1)^\dagger = LXL^\dagger = \mathcal{L}(X) \,. \tag{15}$$

4) The last step to show that every Lorentz transformation has an inverse. Let $\mathcal{L}(X) = LXL^\dagger$ be an arbitrary Lorentz transformation acting on an arbitrary spacetime point X, then, owing to the property that $L\widetilde{L} = 1$, its obvious *inverse* $\mathcal{L}^{-1}$ has the property:

$$\mathcal{L}^{-1}(X) = \widetilde{L}X\widetilde{L}^\dagger \,. \tag{16}$$

So,

$$\mathcal{L}\mathcal{L}^{-1}(X) = (L\widetilde{L})X(L\widetilde{L})^\dagger = X\,, \tag{17}$$

and

$$\mathcal{L}^{-1}\mathcal{L}(X) = (\widetilde{L}L)X(\widetilde{L}L)^\dagger = X\,, \tag{18}$$

where we have used that

$$\widetilde{L}L = 1\,. \tag{19}$$

For a proof of this last result, see the Appendix. And this completes our proof that the set of Lorentz transformations forms a group.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.

4 Appendix

Theorem:

Given that

$$L\widetilde{L} = 1\,, \tag{20}$$

show that

$$\widetilde{L}L = 1\,. \tag{21}$$

Lemma:

We know that the tilde operation performs a space inversion, which is an involution because when applied twice, it undoes what it did the first time, therefore

$$\widetilde{\widetilde{L}} = L\,. \tag{22}$$

Proof of the theorem:

Returning to (20), we multiply through both sides by L on the left and $\widetilde{L}$ on the right, yielding

$$\widetilde{L}L\widetilde{L}L = \widetilde{L}1L = \widetilde{L}L\,. \tag{23}$$

Or, put another way,

$$(\widetilde{L}L)^2 = \widetilde{L}L\,. \tag{24}$$

Therefore, the entity

$$\widetilde{L}L \equiv \mathbf{e} \tag{25}$$

is an idempotent of the spacetime numbers. As such, it's either the trivial idempotent, that is, unity, or else it has nonzero nonscalar parts.

Now, whatever else $\widetilde{L}L$ is, it's the product of two Lorentz transformation (i.e., their representations) and thus is itself a Lorentz transformation (by its representation). But all Lorentz transformations are invertible. However, the idempotent $\mathbf{e}$ is invertible only if it is equal to unity. And that answers that. However, for a more algebraic proof, consider what follows:

Taking the tilde operation on both sides of (25), yields

$$\widetilde{L}\widetilde{\widetilde{L}} = \widetilde{\mathbf{e}}\,. \tag{26}$$

But from (22), we get

$$\widetilde{L}L = \widetilde{\mathbf{e}}\,. \tag{27}$$

So, from this last result and that from (25), we have that $\mathbf{e} = \widetilde{\mathbf{e}}$. This means that $\mathbf{e}$ can have only even parts, say $\mathbf{e} = \alpha + \mathbf{B}$, where α is a scalar and $\mathbf{B}$ is a bivector. Then, inserting this into the equation

$$\mathbf{e}^2 = \mathbf{e}\,, \tag{28}$$

we get

$$(\alpha + \mathbf{B})^2 = \alpha + \mathbf{B}\,. \tag{29}$$

On equating scalar and bivector parts, respectively, we get

$$\begin{array}{ll} \text{Scalar part:} & \alpha^2 - |\,\mathbf{B}\,|^2 = \alpha, \\ \text{Vector part:} & 2\alpha\mathbf{B} = \mathbf{B}. \end{array}$$

If $\mathbf{B} = 0$ then $\alpha = 1$ and then $\mathbf{e} = 1$, therefore

$$\widetilde{L}L = 1\,, \tag{30}$$

which is what we were to show. But what if $\mathbf{B}$ is not identically zero? Then from the vector part we get that $2\alpha = 1$, or $\alpha = 1/2$. But substituting in $1/2$ for α into the scalar part gives the impossible value

$$|\,\mathbf{B}\,|^2 = -1/4\,. \tag{31}$$

Thus, $\mathbf{B}$ is identically zero, which affirms Eq. (21).