Problem 1.4 on Page 589

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1 Problem

On page 589 of NFCM [1], we find problem (1.4): Work out the derivation of (1.38a,b) and (1.39) in complete detail but take care to make it algebraically compact and efficient.

2 Solution to the first part

We begin with the relation we will need. We begin with the knowledge that for all $k = 1, 2, 3$, each B_k is a boost, satisfying the relation

$$B_k^2 = \gamma_k(1 + \mathbf{v}_k/c)\,. \tag{1}$$

We also have that

$$B_k\widetilde{B}_k = 1 \quad \text{(no sum)}\,, \tag{2}$$

and

$$B_3^2 = B_2 B_1^2 B_2\,, \tag{3}$$

We begin with

$$B_3^2 = B_2\gamma_1(1 + \mathbf{v}_1/c)B_2\,. \tag{4}$$

Now, if we break up

$$\mathbf{v}_1 = \mathbf{v}_1^{\|} + \mathbf{v}_1^{\perp}\,, \tag{5}$$

were parallel and prependicualr are relative to the vector $\mathbf{v}_2$, then

$$\mathbf{v}_1^{\|}\mathbf{v}_2 = \mathbf{v}_2\mathbf{v}_1^{\|} \quad \text{and} \quad \mathbf{v}_1^{\perp}\mathbf{v}_2 = -\mathbf{v}_2\mathbf{v}_1^{\perp}\,, \tag{6}$$

where

$$\mathbf{v}_1^{\perp} = \mathbf{v}_1 \wedge \hat{\mathbf{v}}_2\,\hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{v}}_2)\,. \tag{7}$$

This, then, implies that

$$\mathbf{v}_1^{\|}B_2 = B_2\mathbf{v}_1^{\|} \quad \text{and} \quad \mathbf{v}_1^{\perp}B_2 = \widetilde{B}_2\mathbf{v}_1^{\perp}\,, \tag{8}$$

Returning to (4), multiplying it through by c and breaking up $\mathbf{v}_1$ using (5), we have that

$$cB_3^2 = B_2\gamma_1(c + \mathbf{v}_1^{\|} + \mathbf{v}_1^{\perp})B_2\,. \tag{9}$$

Our plan is to 'move the factor of B_2 that's on the right side of the RHS of this last equation to the left', yielding

$$cB_3^2 = B_2\gamma_1(c + \mathbf{v}_1^{\|})B_2 + B_2\gamma_1(\mathbf{v}_1^{\perp})B_2 = B_2^2\gamma_1(c + \mathbf{v}_1^{\|}) + B_2\widetilde{B}_2\gamma_1(\mathbf{v}_1^{\perp})\,. \tag{10}$$

From (2), we get

$$cB_3^2 = B_2^2 \gamma_1 (c + \mathbf{v}_1^{\|}) + \gamma_1 \mathbf{v}_1^{\perp} \,. \tag{11}$$

On expanding B_2^2 and B_3^2

$$\gamma_3 (c + \mathbf{v}_3) = \gamma_2 \gamma_1 (c + \mathbf{v}_2)(c + \mathbf{v}_1^{\|}) + \gamma_1 \mathbf{v}_1^{\perp} \,. \tag{12}$$

Dividing through by $\gamma_1 \gamma_2$, we get

$$\mu (c + \mathbf{v}_3) = (c + \mathbf{v}_2)(c + \mathbf{v}_1^{\|}) + \gamma_2^{-1} \mathbf{v}_1^{\perp} \,, \tag{13}$$

where $\mu = \gamma_3 / \gamma_1 \gamma_2$. Expanding, we get

$$\mu (c + \mathbf{v}_3) = c^2 + c^2 \mathbf{v}_2 + \mathbf{v}_1^{\|} + \mathbf{v}_2 c \mathbf{v}_1^{\|} + \gamma_2^{-1} \mathbf{v}_1^{\perp} \,. \tag{14}$$

Using that $\mathbf{v}_1^{\|} = \mathbf{v}_1 - \mathbf{v}_1^{\perp}$ and that $\mathbf{v}_2 \mathbf{v}_1^{\|} = \mathbf{v}_1 \cdot \mathbf{v}_2$, and use (7), our next step gives us

$$\mu (c + \mathbf{v}_3) = c^2 + c^2 \mathbf{v}_2 + \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + (\gamma_2^{-1} - 1) \hat{\mathbf{v}}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{v}}_2) \,. \tag{15}$$

Taking the scalar part of this, we get

$$\mu = 1 + \mathbf{v}_1 \cdot \mathbf{v}_2 \,, \tag{16}$$

and from the vector part, we get

$$\mu \mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 + (\gamma_2^{-1} - 1) \hat{\mathbf{v}}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{v}}_2) \,. \tag{17}$$

Finally giving us for the vector part:

$$\mathbf{v}_3 = \frac{\mathbf{v}_1 + \mathbf{v}_2 + (\gamma_2^{-1} - 1) \hat{\mathbf{v}}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{v}}_2)}{1 + \mathbf{v}_1 \cdot \mathbf{v}_2} \,. \tag{18}$$

And that gives us the steps to prove (1.38a,b).

The solution to proving (1.39) will be provided at a later date.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.