Notes on Chapter 9 Section 2

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1 Introduction

These notes cover pages 589 to 614 of NFCM [1].

2 Pages 605–8

Let X and X' be on the same hyperbola defined by the locus of points $X\widetilde{X} = 1$. Thus, various points on this hyperbola are connected by a boost L, satisfying

$$X' = LXL = L^2X\,, \tag{1}$$

where $L\widetilde{L} = 1$ and

$$L^2 = \gamma(1 + \mathbf{v}/c) = \gamma(1 + \boldsymbol{\beta})\,, \tag{2}$$

where $\boldsymbol{\beta} \equiv \mathbf{v}/c$. Because $L\widetilde{L} = 1$, we can set $L = e^{\frac{1}{2}\mathbf{a}}$.

On solving (1) for L^2, we get that

$$L^2 = X'\widetilde{X} = e^{\mathbf{a}} = \gamma(1 + \boldsymbol{\beta})\,, \tag{3}$$

where $(\gamma = 1 - \beta^2)^{-1/2}$ and

$$e^{\mathbf{a}} = \cosh a + \hat{\mathbf{a}} \sinh a\,. \tag{4}$$

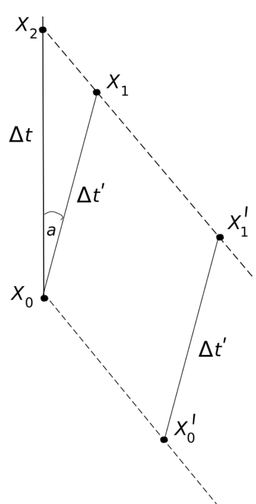

Figure 1. This figure corresponds to Fig. 2.13 (p. 608) in the text.

Taking the scalar part of (3), we get

$$\langle e^{\mathbf{a}} \rangle = \cosh a = \langle \gamma(1+\boldsymbol{\beta}) \rangle = \gamma \,. \tag{5}$$

To setup the algebra of triangle $X_0X_1X_2$, we define the notation:

$$X_{jk} \equiv X_j - X_k \,. \tag{6}$$

Then the vector equation for the triangle is

$$X_{20} = X_{21} + X_{10} \,. \tag{7}$$

On rewriting this as

$$X_{21} = X_{20} - X_{10} \,, \tag{8}$$

and then multiplying through by the respective side's conjugate, we get

$$X_{21}\widetilde{X}_{21} = (X_{20} - X_{10})(\widetilde{X}_{20} - \widetilde{X}_{10}) \,. \tag{9}$$

But X_{21} is lightlike, hence $X_{21}\widetilde{X}_{21} = 0$. Therefore,

$$\begin{aligned} 0 &= (X_{20} - X_{10})(\widetilde{X}_{20} - \widetilde{X}_{10}) \\ &= X_{20}\widetilde{X}_{20} - X_{20}\widetilde{X}_{10} - X_{10}\widetilde{X}_{20} + X_{10}\widetilde{X}_{10} \\ &= X_{20}\widetilde{X}_{20} - 2\langle X_{20}\widetilde{X}_{10} \rangle + X_{10}\widetilde{X}_{10} \\ &= c^2(\Delta t)^2 - 2\gamma\Delta t\Delta t' + c^2(\Delta t')^2 \,. \end{aligned} \tag{10}$$

On dividing through by $c^2(\Delta t)^2$ and setting $D = \Delta t'/\Delta t$, we get

$$D^2 - 2\gamma D + 1 = 0 \,. \tag{11}$$

From this we get

$$D_{\pm} = \gamma \pm \sqrt{\gamma^2 - 1} \,. \tag{12}$$

With some algebra we have that

$$D_{\pm} = \left(\frac{1 \pm \beta}{1 \mp \beta} \right)^{1/2} \,. \tag{13}$$

Thus,

$$\Delta t = D_{\pm}\Delta t' \,. \tag{14}$$

Since $\nu = 1/t$, then

$$\begin{aligned} \nu &= \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \nu' \quad \text{for approaching source} \,, \\ \nu &= \left(\frac{1-\beta}{1+\beta} \right)^{1/2} \nu' \quad \text{for receding source} \,, \end{aligned} \tag{15}$$

which is the case depicted in the figure.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.