Problems 2.1/2.2 on Page 613

P. Reany

September 11, 2021

1 Problem 2.1

On page 613 of NFCM [1], we find problem (2.1): Derive the Doppler formula from the trigonometry of Figure 2.15 (of the text) by first establishing

$$t_1 + t_2 = 2\gamma t' = (t_2 - t_1)\frac{c}{v}\,. \tag{1}$$

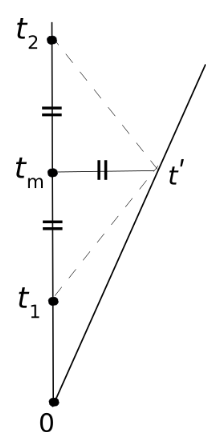

Figure 1. This figure corresponds to Fig. 2.15 (p. 613) in the text.

2 Solution to 2.1

The point t_{m} is the midpoint of the points t_1 and t_2:

$$t_{\mathrm{m}} = \tfrac{1}{2}(t_1 + t_2)\,. \tag{2}$$

We know that t_{m} and t' are related by the following:

$$t_{\mathrm{m}} = \gamma t'\,. \tag{3}$$

The horizontal line segment, which represents $\mathbf{x}$, is equal to $\mathbf{v}t_{\mathrm{m}}$. But by the indicated congruences:

$$vt_{\mathrm{m}} = \frac{c}{2}(t_2 - t_1)\,. \tag{4}$$

On combining these last three equations, we obtain (1).

From (1) we can write

$$t_1 + t_2 = 2\gamma t' \tag{5a}$$
$$t_2 - t_1 = 2\beta\gamma t' \,. \tag{5b}$$

Adding these together, we have that

$$2t_2 = 2\gamma(1+\beta)t' \,, \tag{6}$$

or rather,

$$t_2 = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} t' \,. \tag{7}$$

3 Problem 2.2

Establish the similarity of "lightlike" triangles in Figure 2.15 by proving that

$$t_1 t_2 = t'^2 \,. \tag{8}$$

4 Solution to 2.2

In other words, we are to show that $\triangle(t_1 \mathbf{0} t') \sim \triangle(t' \mathbf{0} t_2)$. Now, if (8) is true, then

$$\frac{t_1}{t'} = \frac{t'}{t_2} \,, \tag{9}$$

and the triangles are similar.

From Eqs. (2) and (4) we have that

$$4t_{\mathrm{m}}^2 = (t_1 + t_2)^2 = t_1^2 + 2t_1 t_2 + t_2^2 \tag{10a}$$
$$4\beta^2 t_{\mathrm{m}}^2 = (t_2 - t_1)^2 = t_1^2 - 2t_1 t_2 + t_2^2 \,. \tag{10b}$$

Subtracting (10b) from (10a) we get

$$4(1-\beta^2)t_{\mathrm{m}}^2 = 4t_1 t_2 \,. \tag{11}$$

We can rewrite this as

$$(\gamma^{-1} t_{\mathrm{m}})^2 = t_1 t_2 \,. \tag{12}$$

Now we use Eq. (3) to get

$$t'^2 = t_1 t_2 \,, \tag{13}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.