Notes on Chapter 9 Section 3

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1 Introduction

These notes cover pages 615 to 632 of NFCM [1].

2 Pages 620–1

The trick here is to go from Eq. (3.26) in the text

$$\frac{dV}{d\tau} = \frac{q}{2mc}(FV + VF^\dagger)\,, \tag{1}$$

to Eq. (3.28)

$$\frac{dU}{d\tau} = \frac{q}{2mc}FU\,. \tag{2}$$

The trick is to replace (3.17)

$$f = f(X, V) \tag{3}$$

with

$$f = f(X, V_0, U) \tag{4}$$

where U is defined by the relation (3.27), that is,

$$V = UV_0U^\dagger\,. \tag{5}$$

To start the transition, we differentiate this last equation to get

$$\frac{dV}{d\tau} = \frac{dU}{d\tau}V_0U^\dagger + UV_0\frac{dU^\dagger}{d\tau}\,. \tag{6}$$

On rearranging (5), we have that

$$V_0U^\dagger = \widetilde{U}V \quad \text{and} \quad UV_0 = V\widetilde{U}^\dagger\,. \tag{7}$$

Substituting these results into (6), we get

$$\frac{dV}{d\tau} = \frac{dU}{d\tau}\widetilde{U}V + V\widetilde{U}^\dagger\frac{dU^\dagger}{d\tau}\,. \tag{8}$$

Comparing this last equation with (1), we get

$$\frac{dU}{d\tau}\widetilde{U} = \frac{q}{2mc}F\,. \tag{9}$$

Multiplying both sides by U on the right and using that

$$U\widetilde{U} = \widetilde{U}U = 1\,, \tag{10}$$

we get that

$$\frac{dU}{d\tau} = \frac{q}{2mc}FU\,, \tag{11}$$

as was required.

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We wish now to go from Eq. (3.26) in the text

$$\frac{dV}{d\tau} = \frac{q}{2mc}(FV + VF^\dagger)\,, \tag{12}$$

to Eq. (3.48)

$$F = LF'\widetilde{L}\,. \tag{13}$$

In other words, if we transform V by

$$V = LV'L^\dagger\,, \tag{14}$$

how, then, does F transform?

First, (12) must also be true in the primed framed, so that

$$\frac{dV'}{d\tau} = \frac{q}{2mc}(F'V' + V'F'^\dagger)\,, \tag{15}$$

On differentiating (16), we get

$$\frac{dV}{d\tau} = L\frac{dV'}{d\tau}L^\dagger\,, \tag{16}$$

On substituting (15) into (16), we get

$$\frac{dV}{d\tau} = L\Big[\frac{q}{2mc}(F'V' + V'F'^\dagger)\Big]L^\dagger\,, \tag{17}$$

which can be massaged into

$$\frac{dV}{d\tau} = \frac{q}{2mc}[LF'V'L^\dagger + LV'F'^\dagger L^\dagger\,]\,. \tag{18}$$

Now, substituting (16) into (12) we have that

$$\frac{dV}{d\tau} = \frac{q}{2mc}(FLV'L^\dagger + LV'L^\dagger F^\dagger)\,. \tag{19}$$

By comparing (18) and (19), we get that

$$LF' = FL \quad \text{and} \quad F'^\dagger L^\dagger = L^\dagger F^\dagger\,. \tag{20}$$

Solving either one of these for F, we get

$$F = LF'\widetilde{L}\,. \tag{21}$$

Lemma: for Eqs. (3.51a) and (3.51b):

For any two nonparallel vectors $\mathbf{G}$ and $\mathbf{u}$ such that $\mathbf{u}^2 = 1$ and $\mathbf{G}_{\shortparallel}$ defined by

$$\mathbf{G}_{\shortparallel} = \mathbf{G} \cdot \mathbf{u}\, \mathbf{u}\,, \tag{22}$$

and

$$\mathbf{G}_{\perp} = \mathbf{G} - \mathbf{G}_{\shortparallel}\,, \tag{23}$$

then

$$\mathbf{G}_{\perp} = \mathbf{u} \times (\mathbf{G} \times \mathbf{u})\,. \tag{24}$$

Proof:

Since

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\mathbf{a} \cdot \mathbf{c} - \mathbf{c}\mathbf{a} \cdot \mathbf{b}\,, \tag{25}$$

then

$$\mathbf{u} \times (\mathbf{G} \times \mathbf{u}) = \mathbf{G}\mathbf{u}^2 - \mathbf{u}\mathbf{u} \cdot \mathbf{G} = \mathbf{G} - \mathbf{G}_{\shortparallel}\,. \tag{26}$$

Hence

$$\mathbf{G}_{\perp} = \mathbf{u} \times (\mathbf{G} \times \mathbf{u})\,. \tag{27}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.