Problem 3.12 on Page 630

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1 Problem 3.1

On page 630 of NFCM [1], we find problem (3.1): According to equation (3.5) the 4-velocities in two inertial systems are related by

$$V = L^{\dagger} V' L \,. \tag{1}$$

or

$$\gamma(c + \mathbf{v}) = L^{\dagger} \gamma'(c + \mathbf{v}')L \,. \tag{2}$$

If the primed frame is related to the unprimed frame by a boost of velocity $\mathbf{u}$, then

$$L^2 = \beta(1 + \mathbf{u}/c) \quad \text{where} \quad \beta = (1 - u^2/c^2)^{-1/2} \,. \tag{3}$$

In this case, derive the "Doppler relation" for material particles

$$\frac{E}{E'} = \frac{\gamma}{\gamma'} = \beta\Big(1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\Big) \,. \tag{4}$$

and the *velocity addition* formula

$$\mathbf{v} = \frac{\mathbf{v}' + \mathbf{u} + \mathbf{u}\times(\mathbf{v}'\times\mathbf{u})c^{-2}(1+\beta^{-1})^{-1}}{1 + \mathbf{u}\cdot\mathbf{v}'/c^2} \,. \tag{5}$$

The rest of the problem is not so difficult.

By the way, Eq. (3.5) in the text is

$$\frac{dX}{d\tau} = L\frac{dX'}{d\tau}L^{\dagger} \,. \tag{6}$$

2 Solution to 3.1

Since L is a boost: $L^{\dagger} = L = L(\mathbf{u})$, $L\widetilde{L} = 1$, and (2) becomes

$$\gamma(c + \mathbf{v}) = L\gamma'(c + \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp})L \,, \tag{7}$$

where $\mathbf{v}'_{\parallel} \equiv \mathbf{v}'\cdot\hat{\mathbf{u}}\,\hat{\mathbf{u}}$. Then

$$\begin{aligned}\gamma(c + \mathbf{v}) &= L^2\gamma'(c + \mathbf{v}'_{\parallel}) + L\widetilde{L}\mathbf{v}'_{\perp} \\ &= \beta(1 + \mathbf{u}/c)\gamma'(c + \mathbf{v}'_{\parallel}) + \mathbf{v}'_{\perp} \,.\end{aligned} \tag{8}$$

Before we separate graded parts, let's simplify a bit.

$$\frac{\gamma}{\gamma'\beta}(c+\mathbf{v}) = (1+\mathbf{u}/c)(c+\mathbf{v}'_\parallel) + \beta^{-1}\mathbf{v}'_\perp \,. \tag{9}$$

Now we extract the scalar part:

$$\frac{\gamma}{\gamma'\beta} = 1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2} \,. \tag{10}$$

And then the vector part:

$$\begin{aligned}
\frac{\gamma}{\gamma'\beta}\mathbf{v} &= \mathbf{u} + \mathbf{v}'_\parallel + \beta^{-1}\mathbf{v}'_\perp \\
&= \mathbf{u} + \mathbf{v}'_\parallel + \mathbf{v}'_\perp + \mathbf{v}'_\perp(\beta^{-1}-1) \\
&= \mathbf{u} + \mathbf{v}' + \hat{\mathbf{u}}\times(\mathbf{v}'\times\hat{\mathbf{u}})(\beta^{-1}-1) \\
&= \mathbf{u} + \mathbf{v}' + u^{-2}\mathbf{u}\times(\mathbf{v}'\times\mathbf{u})(\beta^{-1}-1) \\
&= \mathbf{u} + \mathbf{v}' + c^{-2}\frac{\mathbf{u}\times(\mathbf{v}'\times\mathbf{u})}{u^2/c^2}(\beta^{-1}-1) \,.
\end{aligned} \tag{11}$$

Solving for u^2/c^2 in terms of β, we get

$$\frac{u^2}{c^2} = 1 - \frac{1}{\beta^2} = 1 - \beta^{-2} \,. \tag{12}$$

Therefore,

$$\frac{\beta^{-1}-1}{1-\beta^{-2}} = \frac{\beta^{-1}}{\beta^{-2}}\frac{1-\beta}{\beta^2-1} = -\beta\frac{1-\beta}{1-\beta^2} = -\beta\frac{1}{1+\beta} = -(\beta^{-1}+1)^{-1} \,. \tag{13}$$

Thus we get

$$\frac{\gamma}{\gamma'\beta}\mathbf{v} = \mathbf{u} + \mathbf{v}' + c^{-2}\mathbf{u}\times(\mathbf{v}'\times\mathbf{u})(-(\beta^{-1}+1)^{-1}) \,. \tag{14}$$

On dividing (14) by (10), I get

$$\mathbf{v} = \frac{\mathbf{v}' + \mathbf{u} + \mathbf{u}\times(\mathbf{v}'\times\mathbf{u})c^{-2}[-(1+\beta^{-1})^{-1}]}{1+\mathbf{u}\cdot\mathbf{v}'/c^2} \,, \tag{15}$$

which differs from the text solution (5) by that factor of a negative sign.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.