Problem 3.3 on Page 631

P. Reany

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1 Problem 3.3

On page 631 of NFCM [1], we find problem (3.3): For hyperbolic motion (Figure 3.2), show that the particle's 4-acceleration is tangent to its line of simultaneity at each proper time, and all such lines intersect the event horizon. Show also that the particle's 3-velocity is given by prove that

$$\mathbf{v} = c \tanh \mathbf{g}\tau \,. \tag{1}$$

2 Solution to 3.3

First Part: To show that the 4-acceleration is tangent to its line of simultaneity at each proper time is the same as showing that the 4-acceleration is orthogonal to the 4-velocity, or that

$$\langle \frac{dV}{d\tau} \widetilde{V} \rangle = 0 \,. \tag{2}$$

But by differentiating

$$V\widetilde{V} = c^2 \tag{3}$$

by τ, we get

$$\frac{dV}{d\tau}\widetilde{V} + V\frac{d\widetilde{V}}{d\tau} = 2\langle \frac{dV}{d\tau} \widetilde{V} \rangle = 0 \,. \tag{4}$$

from which follows (2).

Second Part: We begin with Eq. (3.31a) on page 623:

$$X = X_0(1 - e^{\mathbf{g}\tau}) \,, \tag{5}$$

and

$$X_0 = -\mathbf{g}^{-1} V_0 \,. \tag{6}$$

Furthermore, Eq. (5) represents the special situation in which $\mathbf{v}_{\perp 0} = 0$. Now, the only way I was able to arrive at (1), was to assume that $\mathbf{v}_{\| 0} = 0$, as well. In otherwords, that the particle is starting from rest. In this even more special case, we have that

$$X_0 = -\mathbf{g}^{-1} c \,. \tag{7}$$

On differentiating (5) by τ and employing (7), we get that

$$V = c e^{\mathbf{g}\tau} \,. \tag{8}$$

But on substituting in for $V = \gamma(c + \mathbf{v})$ and expanding, we get

$$\gamma(c + \mathbf{v}) = c[\cosh \mathbf{g}\tau + \sinh \mathbf{g}\tau] \,. \tag{9}$$

The scalar part of this equation is

$$\gamma = \cosh \mathbf{g}\tau\,, \tag{10}$$

and the vector part is

$$\gamma \mathbf{v} = c \sinh \mathbf{g}\tau\,. \tag{11}$$

Dividing the latter by the former, we get

$$\mathbf{v} = c \tanh \mathbf{g}\tau\,. \tag{12}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.