Problems 3.4 on Page 631

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September 14, 2021

1 Problem 3.4

On page 613 of NFCM [1], we find problem (3.4): A particle subject to a constant force $\mathbf{f}$ has an initial momentum $\mathbf{p}_0$. Derive the following expression for its velocity.

$$\frac{\mathbf{v}}{c} = \frac{\mathbf{f}t + \mathbf{p}_0}{[\mathbf{f}^2t^2 + 2\mathbf{p}_0 \cdot \mathbf{f}t + \mathbf{p}_0^2 + m^2c^2]^{\frac{1}{2}}}\,. \tag{1}$$

2 Solution to 3.4

Our starting equation is the simple Eq. (3.22) pg. 620:

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}\,, \tag{2}$$

where

$$\mathbf{p} = \gamma m\mathbf{v}\,. \tag{3}$$

So, now we integrate

$$\int_{\mathbf{p}_0}^{\mathbf{p}} d\mathbf{p} = \int_0^t \mathbf{f}\,dt\,, \tag{4}$$

from which we get that

$$\mathbf{p} = \mathbf{f}t + \mathbf{p}_0\,. \tag{5}$$

My plan is to set $\beta = v/c$ and then show that

$$\frac{\mathbf{v}}{c} = \beta\hat{\mathbf{v}}\,. \tag{6}$$

Combining (3) and (5), we get

$$\mathbf{v} = \frac{\mathbf{f}t + \mathbf{p}_0}{\gamma m}\,. \tag{7}$$

Therefore,

$$\hat{\mathbf{v}} = \frac{\mathbf{f}t + \mathbf{p}_0}{|\,\mathbf{f}t + \mathbf{p}_0\,|}\,. \tag{8}$$

Going back to (8), we can write

$$\gamma\frac{\mathbf{v}}{c} = \frac{\mathbf{f}t + \mathbf{p}_0}{mc}\,. \tag{9}$$

Upon we squaring both sides, we get

$$\frac{\beta^2}{1-\beta^2} = \frac{|\,\mathbf{f}t + \mathbf{p}_0\,|^2}{m^2c^2} \equiv K^2\,, \tag{10}$$

where K has been introduced for algebraic convenience. From this we get that

$$\begin{aligned} \beta &= \frac{K}{(1+K^2)^{\frac{1}{2}}} \\ &= \frac{|\,\mathbf{f}t + \mathbf{p}_0\,|/mc}{[(1+|\,\mathbf{f}t + \mathbf{p}_0\,|^2)/m^2c^2]^{\frac{1}{2}}} \\ &= \frac{|\,\mathbf{f}t + \mathbf{p}_0\,|}{[\mathbf{f}^2t^2 + 2\mathbf{p}_0 \cdot \mathbf{f}t + \mathbf{p}_0^2 + m^2c^2]^{\frac{1}{2}}}\,. \end{aligned} \tag{11}$$

Thus

$$\begin{aligned} \frac{\mathbf{v}}{c} &= \beta\hat{\mathbf{v}}\,. \\ &= \frac{|\,\mathbf{f}t + \mathbf{p}_0\,|}{[\mathbf{f}^2t^2 + 2\mathbf{p}_0 \cdot \mathbf{f}t + \mathbf{p}_0^2 + m^2c^2]^{\frac{1}{2}}}\,\frac{\mathbf{f}t + \mathbf{p}_0}{|\,\mathbf{f}t + \mathbf{p}_0\,|} \\ &= \frac{\mathbf{f}t + \mathbf{p}_0}{[\mathbf{f}^2t^2 + 2\mathbf{p}_0 \cdot \mathbf{f}t + \mathbf{p}_0^2 + m^2c^2]^{\frac{1}{2}}}\,, \end{aligned} \tag{12}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.