Notes on Chapter 9, Section 4

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1 Introduction

These notes cover pages 633 to 649 of NFCM [1].

2 Page 647

We begin by establishing the result lying between Eqs. (4.48) and (4.49) in the text. Eq. (4.48) is

$$P_1 = \frac{E'}{c} L^2 - m_2 c\,. \tag{1}$$

Eq. (4.37) is

$$L^2 = \gamma(1+\boldsymbol{\beta}) = \frac{P}{P'} = \frac{m_2c^2 + cP_1}{E'} = \frac{m_2c^2 + E_1 + c\mathbf{p}_1}{E'}\,, \tag{2}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, and where

$$P_k = LP'_k L\,. \tag{3}$$

Eq. (4.38) is a passive Lorentz transformation from the CM frame to the Lab frame.

We also have Eqs. (4.46a)

$$P_3 = U^\dagger P_1 U \tag{4}$$

and (4.46b)

$$U = \widetilde{L}RL = \cos\tfrac{1}{2}\Theta + \widetilde{L}^2\mathbf{i}\sin\tfrac{1}{2}\Theta\,. \tag{5}$$

L^2 is related to $\mathbf{i}$ by Eq. (4.47)

$$\mathbf{i}L^2 = \widetilde{L}^2\mathbf{i}\,. \tag{6}$$

Now, we multiply (1) through by c and replace U by its equivalent in (5)

$$\begin{aligned}
cP_3 &= \big[\cos\tfrac{1}{2}\Theta - \mathbf{i}\widetilde{L}^2\sin\tfrac{1}{2}\Theta\big]\big[E'L^2 - m_2c^2\big]\big[\cos\tfrac{1}{2}\Theta + \widetilde{L}^2\mathbf{i}\sin\tfrac{1}{2}\Theta\big] \\
&= \big[\cos\tfrac{1}{2}\Theta - \mathbf{i}\widetilde{L}^2\sin\tfrac{1}{2}\Theta\big]\big[E'L^2\cos\tfrac{1}{2}\Theta + E'\mathbf{i}\sin\tfrac{1}{2}\Theta - m_2c^2\cos\tfrac{1}{2}\Theta - m_2c^2\widetilde{L}^2\mathbf{i}\sin\tfrac{1}{2}\Theta\big] \\
&= E'L^2\cos^2\tfrac{1}{2}\Theta + E'\mathbf{i}\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta - m_2c^2\cos^2\tfrac{1}{2}\Theta \\
&\quad - \mathbf{i}E'\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta + E'L^2\sin^2\tfrac{1}{2}\Theta - m_2c^2L^4\sin^2\tfrac{1}{2}\Theta \\
&= E'L^2 - m_2c^2[L^4\sin^2\tfrac{1}{2}\Theta + \cos^2\tfrac{1}{2}\Theta] - \mathbf{i}m_2c^2('L^2 - \widetilde{L}^2)\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta \\
&= E'L^2 - m_2c^2\big[\cos^2\tfrac{1}{2}\Theta + L^4\sin^2\tfrac{1}{2}\Theta + \mathbf{i}(L^2 - \widetilde{L}^2)\cos\tfrac{1}{2}\Theta\sin\tfrac{1}{2}\Theta\big]\,.
\end{aligned} \tag{7}$$

Getting from here to Eq. (4.49), which is

$$cP_3 = E_1 - m_2c^2(\gamma^2-1)(1-\cos\Theta) + \frac{m_2c^3}{E'}\mathbf{p}_1\Big(\frac{E'}{m_2c^2} - \gamma + \gamma\cos\Theta + \mathbf{i}\sin\Theta\Big)\,, \tag{8}$$

won't be any easier. Let's begin by presenting a number of results that will be useful in this endeavour. Along with (2) we need:

$$\widetilde{L}^2 = \gamma(1-\boldsymbol{\beta}) = \frac{m_2c^2 + E_1 - c\mathbf{p}_1}{E'} \,. \tag{9}$$

Hence,

$$L^2 - \widetilde{L}^2 = \frac{2c\mathbf{p}_1}{E'} \,. \tag{10}$$

And L^4 is

$$\begin{aligned} L^4 &= \frac{1}{E'^2}[(m_2c^2 + E_1) + c\mathbf{p}_1]^2 \\ &= \frac{1}{E'^2}[(m_2c^2 + E_1)^2 + c^2\mathbf{p}_1^2 + 2(m_2c^2 + E_1)c\mathbf{p}_1] \,. \end{aligned} \tag{11}$$

Also

$$\gamma^2 - 1 = \beta^2\gamma^2 \,, \tag{12}$$

and

$$\gamma = \frac{m_2c^2 + E_1'}{E'} \,, \tag{13}$$

and also

$$\frac{c^2\mathbf{p}_1^2}{E'^2} = \beta^2\gamma^2 = \gamma^2 - 1 \,. \tag{14}$$

Lastly, we need some trigonometric identities:

$$\sin^2 \tfrac{1}{2}\Theta = \frac{1-\cos\Theta}{2} \quad \text{and} \quad \cos^2 \tfrac{1}{2}\Theta = \frac{1+\cos\Theta}{2} \,, \tag{15}$$

and

$$\sin \tfrac{1}{2}\Theta \, \cos \tfrac{1}{2}\Theta = \tfrac{1}{2}\sin\Theta \,. \tag{16}$$

Now, let's dig in:

$$\begin{aligned} cP_3 &= E'L^2 - m_2c^2\big[\cos^2 \tfrac{1}{2}\Theta + L^4 \sin^2 \tfrac{1}{2}\Theta + \mathbf{i}(L^2 - \widetilde{L}^2)\cos\tfrac{1}{2}\Theta \sin\tfrac{1}{2}\Theta\big] \\ &= (m_2c^2 + E' + c\mathbf{p}) - m_2c^2\big[\cos^2 \tfrac{1}{2}\Theta + \frac{1}{E'^2}\big((m_2c^2 + e_1)^2 + c^2\mathbf{p}_1^2 \\ &\quad + 2(m_2c^2 + e_1)c\mathbf{p}_1\big)\sin^2\tfrac{1}{2}\Theta + 2\mathbf{i}\frac{c\mathbf{p}_1}{E'}\cos\tfrac{1}{2}\Theta\sin\tfrac{1}{2}\Theta\big] \\ &= (m_2c^2 + E' + c\mathbf{p}) - m_2c^2\big[\tfrac{1}{2}(1+\cos\Theta) + \frac{1}{E'^2}\big((m_2c^2 + e_1)^2 + c^2\mathbf{p}_1^2 \\ &\quad + 2(m_2c^2 + E_1)c\mathbf{p}_1\big)\tfrac{1}{2}(1-\cos\Theta) + 2\mathbf{i}\frac{c\mathbf{p}_1}{E'}\tfrac{1}{2}\sin\Theta\big] \\ &= E_1 + m_2c^2 + c\mathbf{p} - m_2c^2\Big[\tfrac{1}{2}(1+\cos\Theta) + \gamma^2\tfrac{1}{2}(1-\cos\Theta) \\ &\qquad + \frac{1}{E'^2}\big(c^2\mathbf{p}_1^2 + 2(m_2c^2 + E_1)c\mathbf{p}_1\big)\tfrac{1}{2}(1-\cos\Theta) + \mathbf{i}\frac{c\mathbf{p}_1}{E'}\sin\Theta\Big] \\ &. \end{aligned} \tag{17}$$

We finish by remembering that $\mathbf{i}\mathbf{p}_1 = -\mathbf{p}_1\mathbf{i}$:

$$
\begin{aligned}
cP_3 &= E_1 - m_2c^2\big[-\tfrac{1}{2}(1-\cos\Theta) + \gamma^2\tfrac{1}{2}(1-\cos\Theta) + \frac{c^2\mathbf{p}_1^2}{E'^2}\tfrac{1}{2}(1-\cos\Theta)\big] \\
&\quad + m_2c^3\mathbf{p}_1\Big[\frac{1}{m_2c^2} - \frac{m_2c^2+E_1}{E'^2}(1-\cos\Theta) + \mathbf{i}\frac{\sin\Theta}{E'}\Big] \\
&= E_1 - m_2c^2\big[-\tfrac{1}{2}(1-\cos\Theta)\Big(\gamma^2 - 1 + \frac{c^2\mathbf{p}_1^2}{E'^2}\Big)\big] \\
&\quad + \frac{m_2c^3\mathbf{p}_1}{E'}\Big[\frac{E'}{m_2c^2} - \gamma\big(1-\cos\Theta\big) + \mathbf{i}\sin\Theta\Big] \\
&= E_1 - m_2c^2(\gamma^2-1)(1-\cos\Theta) + \frac{m_2c^3}{E'}\mathbf{p}_1\Big(\frac{E'}{m_2c^2} - \gamma + \gamma\cos\Theta + \mathbf{i}\sin\Theta\Big)\,. \qquad (18)
\end{aligned}
$$

Now we state a few more results we will need:

$$
E' = c^2[(m_1+m_2)^2 + 2m_2K_1/c^2]^{1/2}\,, \qquad (19)
$$

and

$$
\gamma^2 - 1 = \frac{(2m_1c^2+K_1)K_1}{E'^2} = \frac{(2m_1c^2+K_1)K_1}{c^4[(m_1+m_2)^2+2m_2K_1/c^2]}\,. \qquad (20)
$$

And for good measure, we also have that

$$
cP_k = E_k + c\mathbf{p}_k \quad \text{and} \quad K_k = (\gamma-1)m_kc^2\,. \qquad (21)
$$

Now we separate the scalars from the vectors of (18). The scalar part yields

$$
E_3 = E_1 - m_2c^2(\gamma^2-1)(1-\cos\Theta)\,, \qquad (22a)
$$

and the vector part yields

$$
c\mathbf{p}_3 = \frac{m_2c^3}{E'}\mathbf{p}_1\Big(\frac{E'}{m_2c^2} - \gamma + \gamma\cos\Theta + \mathbf{i}\sin\Theta\Big)\,. \qquad (22b)
$$

Eq. (22a) can be rewritten as

$$
\begin{aligned}
(\gamma-\gamma_3)m_1c^2 &= m_2c^2(\gamma^2-1)(1-\cos\Theta) \\
&= m_2c^2\frac{(2m_1c^2+K_1)K_1}{c^4[(m_1+m_2)^2+2m_2K_1/c^2]}(1-\cos\Theta) \\
&= m_2\frac{(2m_1+K_1/c^2)K_1}{[(m_1+m_2)^2+2m_2K_1/c^2]}(1-\cos\Theta)\,, \qquad (23)
\end{aligned}
$$

where we used (20). Now,

$$
\begin{aligned}
\frac{\Delta K}{K_1} \equiv \frac{K_1-K_3}{K_1} &= \frac{(\gamma-1)m_1c^2 - (\gamma_3-1)m_1c^2}{K_1} \\
&= \frac{(\gamma-\gamma_3)m_1c^2}{K_1} \\
&= \frac{m_2(2m_1+K_1/c^2)(1-\cos\Theta)}{(m_1+m_2)^2+2m_2K_1/c^2}\,. \qquad (24)
\end{aligned}
$$

From the vector part we need to show that

$$
\tan\theta = \frac{\sin\Theta}{\gamma(\alpha+\cos\Theta)}\,, \qquad (25)
$$

where

$$\alpha \equiv \frac{E'}{\gamma m_2 c^2} - 1\,, \tag{26}$$

and θ is the angle between the incoming particle direction and the scattering direction, as seen in the lab frame. The cosine of this angle is given as $\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3$. On multiplying (27) through by $\mathbf{p}_1^{-1}$ and simplifying, we get

$$\mathbf{p}_1^{-1}\mathbf{p}_3 = \frac{m_2 c^2}{E'}\Big(\frac{E'}{m_2 c^2} - \gamma + \gamma\cos\Theta + \mathbf{i}\sin\Theta\Big)\,. \tag{27}$$

The scalar part of this gives us

$$\frac{p_3}{p_1}\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3 = \frac{p_3}{p_1}\cos\theta = \frac{m_2 c^2}{E'}\Big(\frac{E'}{m_2 c^2} - \gamma + \gamma\cos\Theta\Big)\,. \tag{28}$$

The bivector part of this gives us

$$\frac{p_3}{p_1}\sin\theta = \frac{m_2 c^2}{E'}\sin\Theta\,. \tag{29}$$

On dividing this last equation by the one before it, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\Theta}{\dfrac{E'}{m_2 c^2} - \gamma + \gamma\cos\Theta}\,. \tag{30}$$

Employing α we defined above, yields the text equation (4.52):

$$\tan\theta = \frac{\sin\Theta}{\gamma(\alpha + \cos\Theta)}\,. \tag{31}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.