Notes on Chapter 9, Section 5

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1 Introduction

These notes cover pages 650 to 660 of NFCM [1].

2 The spinor equation for rotations, Fermi-Walker Transport

Pages 650–652.

We begin with three orthonormal vectors $\{\mathbf{e}_i\}$ attached to the point particle. The equation of motion for these axes is given by

$$\frac{d\mathbf{e}_k}{dt} = \boldsymbol{\omega} \times \mathbf{e}_k = -\frac{i}{2}(\boldsymbol{\omega}\mathbf{e}_k - \mathbf{e}_k\boldsymbol{\omega})\,. \tag{1}$$

We choose instead to introduce the spinor R to solve this problem, where

$$\frac{dR}{dt} = -\tfrac{1}{2}i\boldsymbol{\omega}R \quad \text{and} \quad \mathbf{e}_k = R\boldsymbol{\sigma}_k R^\dagger\,. \tag{2}$$

where the $\boldsymbol{\sigma}$'s are fixed in the lab frame and $R^\dagger R = 1$, as usual. We need to adapt this procedure to special relativity. In this case, we'll adopt 4-vectors, a scalar plus a vector. For our 4-velocity V we have

$$V = \gamma(c + \mathbf{v}) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \tag{3}$$

and $\beta = v/c$ and for future reference, $\boldsymbol{\beta} = \mathbf{v}/c$.

Let the rest frame of the particle be the primed system. Naturally, in this frame the velocity of the particle $\mathbf{v}'$ is zero, therefore $V' = c$. Now we'll perform a Lorentz boost from the rest frame of the particle to the lab frame, given by

$$V = LV'L = cL^2 = \gamma(c + \mathbf{v})\,, \tag{4}$$

according to (2.23) on pg. 605 of the text. We also have that $L = L^\dagger$ and $L\widetilde{L} = 1$. We also require that this same boost connects from the instanteous rest frame's version of the particle's principal axes $\mathbf{e}_k$ to what the observer in the lab would regard them as E_k

$$E_k = L\mathbf{e}_k L\,. \tag{5}$$

But

$$\mathbf{e}_k = R\boldsymbol{\sigma}_k R^\dagger\,, \tag{6}$$

therefore,

$$E_k = (LR)\mathbf{e}_k(LR)^\dagger\,. \tag{7}$$

Let $U \equiv LR$, so then

$$E_k = U\mathbf{e}_k U^\dagger\,. \tag{8}$$

Time to show that $U\widetilde{U} = 1$. First, we know that $R = \langle R \rangle_2$ and that

$$(MN)^\sim = \widetilde{N}\widetilde{M}\,, \tag{9}$$

from (1.16) pg. 580.

$$U\widetilde{U} = LR(LR)^\sim = LRR^\dagger\widetilde{L} = L\widetilde{L} = 1\,. \tag{10}$$

Okay, and

$$\begin{aligned} UV'U^\dagger &= cUU^\dagger \\ &= cLR(LR)^\dagger \\ &= cLRR^\dagger L^\dagger \\ &= cLL^\dagger \\ &= cL^2 = V\,, \end{aligned} \tag{11}$$

which we rewrite as

$$V = UV'U^\dagger\,. \tag{12}$$

Therefore, the spinor U contains all we need to calculate both the Lorentz boost and the rotation of the frame, but how to solve for it?

Differentiating $U\widetilde{U} = 1$ by proper time τ, we get $\dot{U}\widetilde{U} + U\dot{\widetilde{U}} = 0$, or

$$U\dot{\widetilde{U}} = -\dot{U}\widetilde{U}\,. \tag{13}$$

Next, we introduce the multivector Ω, given by

$$\Omega \equiv 2\dot{U}\widetilde{U}\,. \tag{14}$$

Then

$$\widetilde{\Omega} = 2U\dot{\widetilde{U}} = 2U\widetilde{\dot{U}} = -2\dot{U}\widetilde{U} = -\Omega\,. \tag{15}$$

Hence, Ω may contain only vector and/or bivector parts.

$$\Omega = \boldsymbol{\alpha} + i\boldsymbol{\beta}\,, \tag{16}$$

where both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors. Together, the six components of these two vectors give us six degrees of freedom. On rewriting (15), we get the spinor equation of motion:

$$\dot{U} = \tfrac{1}{2}\Omega U\,. \tag{17}$$

On differentiating (12), we get

$$\begin{aligned} \dot{V} &= \dot{U}V'U^\dagger + UV'\dot{U}^\dagger \\ &= \tfrac{1}{2}\Omega V'U^\dagger + \tfrac{1}{2}V'U^\dagger\Omega^\dagger \\ &= \tfrac{1}{2}\left[\Omega V + V\Omega^\dagger\right], \end{aligned} \tag{18}$$

which is (5.10a). Similarly, on differentiating (8) and using (17), we get

$$\begin{aligned} \dot{E}_k &= \dot{U}\boldsymbol{\sigma}_k U^\dagger + U\boldsymbol{\sigma}_k\dot{U}^\dagger \\ &= \tfrac{1}{2}\Omega\boldsymbol{\sigma}_k U^\dagger + \tfrac{1}{2}\boldsymbol{\sigma}_k U^\dagger\Omega^\dagger \\ &= \tfrac{1}{2}\left[\Omega E_k + E_k\Omega^\dagger\right], \end{aligned} \tag{19}$$

which is (5.10b).

We will see that Ω contains all the dynamical information we need, but we must delineate the separate roles of "internal" vs "external" dynamics, by making the "split"

$$\Omega = \Omega_+ + \Omega_- \,, \tag{20}$$

where

$$\Omega_\pm = \tfrac{1}{2}(\Omega \pm c^{-2} V\Omega^\dagger \widetilde{V}) \,. \tag{21}$$

In particular,

$$\Omega_+ = \frac{c^{-2}}{2}(\Omega V + V\Omega^\dagger) = c^{-2}\dot{V}\widetilde{V} \,. \tag{22}$$

It's now time to show that Ω in the $\dot{V}$ equation in (23) can be replaced by Ω_+ to get

$$\dot{V} = \tfrac{1}{2}\big[\Omega_+ V + V\Omega_+^\dagger\big] \,, \tag{23}$$

which means that Ω_- describes the rotational motion.

Let's define $2\dot{V}_+ = \Omega_+ V + V\Omega_+^\dagger$, remembering that $V\widetilde{V} = \widetilde{V}V = c^2$, then

$$\begin{aligned}
2\dot{V}_+ &= \Omega_+ V + V\Omega_+^\dagger \\
&= \tfrac{1}{2}\big(\Omega + \frac{1}{c^2}V\Omega^\dagger\widetilde{V}\big)V + V\tfrac{1}{2}\big(\Omega + \frac{1}{c^2}V\Omega^\dagger\widetilde{V}\big)^\dagger \\
&= \tfrac{1}{2}(\Omega V + V\Omega^\dagger) + \tfrac{1}{2}V\big(\Omega^\dagger + \frac{1}{c^2}\widetilde{V}^\dagger\Omega V^\dagger\big) \\
&= \tfrac{1}{2}(\Omega V + V\Omega^\dagger) + \tfrac{1}{2}(\Omega V + V\Omega^\dagger) \\
&= \Omega V + V\Omega^\dagger \\
&= 2\dot{V} \,,
\end{aligned} \tag{24}$$

which establishes (23).

To complete Eq. (5.12) pg. 652, remembering that $\dot{V} = \langle\, \dot{V} \,\rangle_{0+1}$:

$$\langle\, \Omega_+ V \,\rangle_{0+1} = \langle\, \Omega_+ V \,\rangle_{0+1}^\dagger = \langle\, V\Omega_+^\dagger \,\rangle_{0+1} \,, \tag{25}$$

hence

$$\dot{V} = \Omega_+ V = V\Omega_+^\dagger \,. \tag{26}$$

If we insist that the motion be without rotation, then $\Omega_- = 0$ and (22) becomes

$$\Omega = c^{-2}\dot{V}\widetilde{V} \,. \tag{27}$$

This motion is known as *Fermi-Walker Transport*, in which case, (5.10b) becomes

$$\dot{E}_k = \frac{1}{c^2}\big[\dot{V}\widetilde{V}E_k + E_k\widetilde{V}\dot{V}\big] \,, \tag{28}$$

and the text adds this disclaimer (pg. 652): "This equation is needed for each E_k to maintain the orthogonality condition $\langle\, E_k\widetilde{V} \,\rangle = 0$ as it moves along the particle history."

3 The Thomas Precession

We return to the factorization of the text equation (5.6):

$$U = LR\,. \tag{29}$$

On differentiating this and employing Equation (2), we get

$$\dot{U} = \dot{L}R + L\dot{R} = \tfrac{1}{2}(2\dot{L}\widetilde{L} - Li\gamma\boldsymbol{\omega}\widetilde{L})U\,, \tag{30}$$

and we remember that

$$\dot{R} = \gamma\frac{dR}{dt} = -\tfrac{1}{2}i\gamma\boldsymbol{\omega}R\,. \tag{31}$$

Now, we can compare this last equation with Eq. (17) to find a form for Ω:

$$\Omega = 2\dot{L}\widetilde{L} - Li\gamma\boldsymbol{\omega}\widetilde{L}\,, \tag{32}$$

which is Eq. (5.15) of the text. From this we get

$$i\gamma\boldsymbol{\omega} = \widetilde{L}\dot{L} - \widetilde{L}\Omega L\,. \tag{33}$$

Clearly we will need some useful expression for $\widetilde{L}\dot{L}$ in terms of velocity and acceleration before we can solve for Ω_+. By the way, we get an expression for Ω_+ from Eq. (26):

$$\Omega_+ = c^{-2}\dot{V}\widetilde{V}\,. \tag{34}$$

Let's start here. Since $V = \gamma(c + \mathbf{v})$,

$$\begin{aligned}
\dot{V}\widetilde{V} &= [\dot{\gamma}(c+\mathbf{v}) + \gamma\dot{\mathbf{v}}]\gamma(c-\mathbf{v}) \\
&= \dot{\gamma}\gamma(c^2 - \mathbf{v}^2) + \gamma^2\dot{\mathbf{v}}(c-\mathbf{v}) \\
&= \gamma\{\dot{\gamma}(c^2-\mathbf{v}^2) - \gamma\dot{\mathbf{v}}\cdot\mathbf{v} + \gamma(c\dot{\mathbf{v}} + \mathbf{v}\wedge\dot{\mathbf{v}})\}\,.
\end{aligned} \tag{35}$$

But since $V\widetilde{V} = c^2$, then its derivative gives us

$$\langle\, \dot{V}\widetilde{V}\,\rangle = 0\,. \tag{36}$$

Therefore, the scalar part of (35) is zero:

$$0 = \gamma\{\dot{\gamma}(c^2 - \mathbf{v}^2) - \gamma\dot{\mathbf{v}}\cdot\mathbf{v}\}\,. \tag{37}$$

From this we get that

$$\dot{\gamma} = c^{-2}\gamma^3\dot{\mathbf{v}}\cdot\mathbf{v}\,. \tag{38}$$

A result we'll be needing soon follows from this last equation:

$$c^{-1}\langle\, \dot{V}\,\rangle = \dot{\gamma}\,. \tag{39}$$

Anyway, our resulting form for $\dot{V}\widetilde{V}$ is

$$c^2\Omega_+ = \dot{V}\widetilde{V} = \gamma^2(c\dot{\mathbf{v}} + \mathbf{v}\wedge\dot{\mathbf{v}}) = \gamma^2(c\dot{\mathbf{v}} + i\mathbf{v}\times\dot{\mathbf{v}})\,, \tag{40}$$

which is Eq. (5.19) in the text.

As for getting to the equation at the bottom of page 653 in the text, starting with Eq. (5.20):

$$L = (c^{-1}V)^{1/2} = \frac{1 + c^{-1}V}{[2(1+\gamma)]^{1/2}}\,, \tag{41}$$

all I can say in its defense is that I did it my way. Let

$$L = \rho(c + V) \quad \text{where} \quad \rho \equiv \frac{c^{-1}}{[2(1+\gamma)]^{1/2}}\,, \tag{42}$$

Okay, so

$$\begin{aligned}\dot{L}\widetilde{L} &= [\,\dot{\rho}(c+V) + \rho\dot{V}][\rho(c+\widetilde{V})\,] \\ &= \rho^2[\,\dot{\rho}\rho^{-1}(c+V) + \dot{V}][(c+\widetilde{V})] \\ &= \rho^2[\,\dot{\rho}\rho^{-1}(c+V)(c+\widetilde{V}) + \dot{V}c + \dot{V}\widetilde{V}] \\ &= \rho^2[\,\dot{\rho}\rho^{-1}[c^2 + c(V+\widetilde{V}) + V\widetilde{V}] + \dot{V}c + \dot{V}\widetilde{V}]\,.\end{aligned} \tag{43}$$

But $c(V + \widetilde{V}) = 2c^2\gamma$ and $V\widetilde{V} = c^2$, therefore

$$\begin{aligned}\dot{L}\widetilde{L} &= \rho^2[\dot{\rho}\rho^{-1}(c^2 + 2c^2\gamma + c^2) + c^{-1}\dot{V} + c^{-2}\dot{V}\widetilde{V}] \\ &= \rho^2[2\dot{\rho}\rho^{-1}(c^2(1+\gamma)) + c^{-1}\dot{V} + c^{-2}\dot{V}\widetilde{V}]\,.\end{aligned} \tag{44}$$

It's funny how our problems repeat: We already found $\dot{V}\widetilde{V}$ and now we're searching for $\dot{L}\widetilde{L}$, and in the middle of it, we now need to find $2\dot{\rho}\rho^{-1}$. I'll just present the answer:[1]

$$2\dot{\rho}\rho^{-1} = -\frac{\dot{\gamma}}{1+\gamma}\,. \tag{45}$$

Continuing,

$$\begin{aligned}\dot{L}\widetilde{L} &= \frac{1}{2(1+\gamma)}[-\dot{\gamma} + c^{-1}\dot{V} + c^{-2}\dot{V}\widetilde{V}] \\ &= \frac{1}{2(1+\gamma)}[-c^{-1}\langle\,\dot{V}\,\rangle + c^{-1}\dot{V} + c^{-2}\dot{V}\widetilde{V}] \quad \text{(using (39))} \\ &= \frac{1}{2(1+\gamma)}[c^{-1}\langle\,\dot{V}\,\rangle_1 + c^{-2}\dot{V}\widetilde{V}]\,,\end{aligned} \tag{46}$$

and this can be rewritten as

$$\begin{aligned}2\dot{L}\widetilde{L} &= \frac{c\langle\,\dot{V}\,\rangle_1 + \dot{V}\widetilde{V}}{c^2(1+\gamma)} \\ &= \frac{c\dot{\gamma}\mathbf{v} + c\gamma\dot{\mathbf{v}} + \gamma^2(c\dot{\mathbf{v}} + \mathbf{v}\wedge\dot{\mathbf{v}})}{c^2(1+\gamma)} \\ &= \frac{\gamma^2(c\dot{\mathbf{v}} + \mathbf{v}\wedge\dot{\mathbf{v}}) + c(\dot{\gamma}\mathbf{v} + \gamma\dot{\mathbf{v}})}{c^2(1+\gamma)}\,,\end{aligned} \tag{47}$$

and this brings us to the equation at the bottom of page 653.

The vector part of (47) is

$$\langle\,2\dot{L}\widetilde{L}\,\rangle_1 = \frac{\gamma^2\dot{\mathbf{v}} + \dot{\gamma}\mathbf{v} + \gamma\dot{\mathbf{v}}}{c(1+\gamma)} = \frac{\dot{\gamma}\mathbf{v}}{c(1+\gamma)} + \frac{\gamma}{c}\dot{\mathbf{v}}\,. \tag{48}$$

[1] I found it convenient to differentiate ρ^2.

The bivector part of (47) is

$$\langle 2\dot{L}\widetilde{L}\rangle_2 = \frac{\gamma^2 i\mathbf{v}\times\dot{\mathbf{v}}}{c^2(1+\gamma)}\,. \tag{49}$$

For the case of Fermi-Walker transport, we get (see page 654)

$$\boldsymbol{\omega} = \boldsymbol{\omega}_+ = \frac{\gamma^2}{c^2(1+\gamma)}\mathbf{v}\times\dot{\mathbf{v}}\,, \tag{50}$$

where $\boldsymbol{\omega}_+$ is the Thomas Precession. As the text says: "The Thomas precession can be interpreted as a rotation induced by shifting from of these inertial systems to another."

4 Particles with an Intrinsic Magnetic Moment (pg. 654–660)

I hope to do this subsection by the end of the year.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.