Nth-Order Linear DEQs in Geometric Algebra

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Abstract

A couple significant results from the theory of nth-order linear DEQs are produced here by the convenience of geometric algebra, such as Abel's Identity and the Variation of Parameters.

1 Introduction

This paper is based on my earlier paper that I believe was published in the journal *Advances in Applied Clifford Algebras* in 1994–1995, which I reference here [1]. And that paper was based somewhat on the earlier paper [2].

I've lost all but my preprint version of the paper. Judging by the *AACA* website, they seem to have 'lost' the first six years of their journal, as their records seem to go back only so far as 1997, though the journal started in 1991. All my papers published through the *AACA* were published in those missing years. In any case, this paper is rewritten completely from that published paper.

Of note in this paper are the presentations of the nth-order forms of Abel's result and the Variation of Parameters from their familar second-order forms.

The form we will adopt for our differential equations to be used is

$$\frac{d^n}{dx^n}Y + a_1(x)\frac{d^{n-1}}{dx^{n-1}}Y + \cdots + a_{n-1}(x)\frac{d}{dx}Y + a_n(x)Y = b(x)\,, \tag{1}$$

where x is real valued, and $a_1(x)$, $a_2(x)$,…, $b(x)$ are real- or complex-valued. Following convention, we can define the linear operator[1]

$$L \equiv \frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)\,, \tag{2}$$

so that (1) then takes the form

$$L(Y) = b(x)\,. \tag{3}$$

[1]See Appendix 1 for a basic description of a linear operator.

2 The Homogenous Equation

Now, Y in (3) is not just one solution, but rather $n+1$ solutions, beginning with n solutions

$$\{y_1, y_2, \ldots, y_n\} \tag{4}$$

for the *homogeneous* version of Equation (3), given by[2]

$$L(Y) = 0\,, \tag{5}$$

and one more solution ψ_0 to (3), called the *particular solution*, which we'll deal with later. The solutions (4) are linearly independent if the *Wronskian* W (to be defined below) is different from zero. (See Appendix 0 to learn about the importance of the Wronskian in linear differential equations.)

Now we will build a vector out of the solutions to our general nth-order linear homogeneous equation. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ be an orthonormal basis for our space. Consider

$$\mathbf{y} \equiv y_1\sigma_1 + y_2\sigma_2 + \cdots + y_n\sigma_n\,. \tag{6}$$

Up to this point, L has operated on scalar functions, but now we generalize L to operate on vectors. In particular, consider $L(\mathbf{y})$ as a linear operator on C^n, that is : $C^n \to C^n$:

$$\begin{aligned} L(\mathbf{y}) &= L(y_1\sigma_1 + y_2\sigma_2 + \cdots + y_n\sigma_n) \\ &= L(y_1)\sigma_1 + L(y_2)\sigma_2 + \cdots + L(y_n)\sigma_n \\ &= 0\sigma_1 + 0\sigma_2 + \cdots + 0\sigma_n \\ &= 0\,, \end{aligned} \tag{7}$$

since $L(y_i) = 0$ for $i \in [1, 2, \cdots, n]\,.$

Let C_n be the geometric algebra over the complex vector space C^n. We define the *Wronski pseudovector*[3] $\mathbf{P}$ by

$$\mathbf{P} \equiv \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)}\,, \tag{8}$$

where the primes denote differentiation by x. We also define the *Wronski* $\mathbf{W}$ by

$$\mathbf{W} \equiv \mathbf{P} \wedge \mathbf{y}^{(n-1)}\,. \tag{9}$$

Thus the Wronski is a pseudoscalar for our n-dimensional vector space.[4] Furthermore, it's clear that[5]

$$\mathbf{W}' = \mathbf{P} \wedge \mathbf{y}^{(n)}\,. \tag{10}$$

[2]This paper does not concern itself with how one solves for these n solutions to the homogeneous equation; it merely takes their existence for granted.

[3]Warning: The name *Wronski pseudovector* does not exist in the literature.

[4]A **pseudoscalar** for a vector space of dimension n is a nonzero wedge of n vectors of the space.

[5]See Appendix 2 for the calculations.

Let $\mathbf{I}$ be the unit pseudoscalar for C_n, and let $\mathbf{I}^\dagger$ be the reversion of $\mathbf{I}$ so that $\mathbf{I}\mathbf{I}^\dagger = \mathbf{I}^\dagger\mathbf{I} = 1$. The Wronskian W is defined as the determinant of the Wronski

$$W = \mathbf{W}\mathbf{I}^\dagger \,. \tag{11}$$

Lastly, we define the *Wronski vector*

$$\widetilde{\mathbf{P}} \equiv \mathbf{I}^\dagger\mathbf{P} \,. \tag{12}$$

From $L(\mathbf{y}) = 0$ we can immediately form the equation

$$\langle \mathbf{P}L(\mathbf{y})\rangle_n \;= 0 \tag{13}$$

from which we get

$$W' + a_1(x)W = 0 \,, \tag{14}$$

where W is just a scalar function of x. (See Appendix 3 for the calculations.) From (14) we get Abel's Identity for the Wronskian

$$W = c_1 \exp\left\{-\int a_1(x)dx\right\} \,. \tag{15}$$

Let's look at the Wronskian in two dimensions. Let $\mathbf{y} = f\sigma_1 + g\sigma_2$. Then $\mathbf{y}' = f'\sigma_1 + g'\sigma_2$ and the Wronski pseudoscalar is

$$\mathbf{W} = \mathbf{y}\wedge\mathbf{y}' = (f\sigma_1 + g\sigma_2)\wedge(f'\sigma_1 + g'\sigma_2) = (f'g - fg')\sigma_1\wedge\sigma_2 \,. \tag{16}$$

Now, we have a lot of choice for the form of the pseudoscalar $\mathbf{I}$ we need to construct $\mathbf{I}^\dagger$. Let's make it easy on ourselves and choose $\mathbf{I} = \sigma_1\wedge\sigma_2$, then

$$W = \mathbf{W}\mathbf{I}^\dagger = (f'g - fg')(\sigma_1\wedge\sigma_2)\mathbf{I}^\dagger = f'g - fg' \,. \tag{17}$$

More generally, we can define the pseudoscalar quantity

$$\mathbf{W}_k \equiv \langle\, \mathbf{y}\mathbf{y}'\cdots\mathbf{y}^{(k-1)}\mathbf{y}^{(n)}\mathbf{y}^{(k+1)}\cdots\mathbf{y}^{(n-1)}\rangle_n \,, \tag{18}$$

and the scalar quantity

$$W_k \equiv \det(\mathbf{W}_k) = \mathbf{W}_k\mathbf{I}^\dagger \,. \tag{19}$$

Now, multiply $L(\mathbf{y}) = 0$ on the left by $\mathbf{y}\mathbf{y}'\cdots\mathbf{y}^{(k-1)}$ and on the right by $\mathbf{y}^{(k+1)}\cdots\mathbf{y}^{(n-2)}\mathbf{y}^{(n-1)}$, to yield

$$\mathbf{W}_k + a_k(x)\mathbf{W} = 0 \,. \tag{20}$$

By multiplying on the right by $\mathbf{I}^\dagger$ and solving for $a_k(x)$, we get

$$a_k(x) = -W/W_k \,. \tag{21}$$

For a comparison see [3] p. 463–467. (See Appendix 4 for the calculations.)

3 Nonhomogeneous Equation: The Variation of Parameters

Problem: Solve for the particular solution to $L(Y) = b(x)$ using, if possible, the n linearly independent solutions to the homogeneous equation $\{y_1, y_2, \ldots, y_n\}$.

Solution: The particular solution we seek $\phi_0(x)$ (which must not be a linear combination of the homogeneous solutions), must satisfy

$$L(\psi_0) = b(x)\,. \tag{22}$$

Let $\mathbf{y}$ be a vector such that $L(\mathbf{y}) = 0$ as before in (6). We shall attempt a particular solution ansatz in the form

$$\psi_0(x) = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n\,, \tag{23}$$

where the v_i's are functions of x. But we have to reason our way to even suggest such an ansatz as (23). Remembering that L is a linear operator over the space of complex-valued functions, then for n complex numbers c_i

$$L(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) = c_1 L(y_1) + c_2 L(y_2) + \cdots + c_n L(y_n) = 0\,. \tag{24}$$

The idea then is that if we let the coefficients of the y_i's be functions of x, to be determined, then maybe the particular solution can be found from (23). Forming the vector

$$\mathbf{v} = v_1\sigma_1 + v_2\sigma_2 + \cdots + v_n\sigma_n\,, \tag{25}$$

and the spinor[6]

$$\psi = \mathbf{v}\mathbf{y}\,, \tag{26}$$

then

$$\psi_0 = \mathbf{v}\cdot\mathbf{y} = \langle \mathbf{v}\mathbf{y}\rangle\,, \tag{27}$$

and we obtain the highly useful result that[7]

$$L(\psi_0) = \langle L(\mathbf{v}\mathbf{y})\rangle = b(x)\,. \tag{28}$$

Okay, we now have the n variables $v_i(x)$ to solve for, but only one constraint equation on them, which is given by (28). Experience tells us that we're going to need $n-1$ more constraint equations to be able to solve for all the $v_i(x)$'s, and we just need to invent them.

To that end we set the constraints[8]

$$\langle\, \mathbf{v}'\mathbf{y}\,\rangle = \langle \mathbf{v}'\mathbf{y}'\rangle = \cdots = \langle\, \mathbf{v}'\mathbf{y}^{(n-2)}\rangle = 0\,, \tag{29}$$

[6] Please don't let this name scare you.

[7] As a general rule, when algebraically solving for variables in geometric algebra, it's usually useful to convert dot products and wedge products into geometric products.

[8] Setting these particular constraints is probably not obvious to the reader, however, they are suggested as generalizations of the constraints set down in the solution to the Variation of Parameters solution in second-degree linear differential equations, which is $\mathbf{v}'\cdot\mathbf{y} = 0$.

and from these we get the relation

$$\mathbf{v}' \cdot \mathbf{P} = 0\,. \tag{30}$$

(See Appendix 5 for this calculation.)

By differentiating the equations in (29), we get the other constraints we need. For example, we get

$$\langle\, \mathbf{v}''\mathbf{y} + \mathbf{v}'\mathbf{y}' \,\rangle \equiv 0\,. \tag{31}$$

However, from (29) we know that $\langle \mathbf{v}'\mathbf{y}' \rangle = 0$, therefore (31) becomes just $\langle \mathbf{v}''\mathbf{y} \rangle = 0$. If we continue to take derivatives of these constraints we find that all terms with $\mathbf{v}$ to first- or higher-order derivatives are zero except for the term in the derivatives of $\mathbf{y}$, $\langle \mathbf{v}'\mathbf{y}^{(n-1)} \rangle$. By expanding L in (28) all those terms with undifferentiated $\mathbf{v}$ add up to $\langle \mathbf{v}L(\mathbf{y}) \rangle = 0$ (because $L(\mathbf{y}) = 0$), leaving only $\langle\, \mathbf{v}'\mathbf{y}^{(n-1)} \rangle$.[9]

Therefore (28) reduces to

$$\langle \mathbf{v}'\mathbf{y}^{(n-1)} \rangle = b(x)\,. \tag{32}$$

Thus we get from (32):

$$\langle \mathbf{v}'\mathbf{P}^{-1}\mathbf{P}\mathbf{y}^{(n-1)} \rangle = \mathbf{v}'\mathbf{P}^{-1}\mathbf{I}W = b(x)\,. \tag{33}$$

where we used that $\mathbf{v}' \cdot \mathbf{P}^{-1} = 0$ implies that $\mathbf{v}' \wedge \mathbf{P}^{-1} = \mathbf{v}'\,\mathbf{P}^{-1}$.

Solving for $\mathbf{v}'$ and integrating gives

$$\mathbf{v} = \int \frac{b(x)\widetilde{\mathbf{P}}\,dx}{W}\,, \tag{34}$$

where we used (12). Thus

$$\psi_0 = \mathbf{v}(x) \cdot \mathbf{y}(x) = \int^x \frac{\Delta(t,x)}{W(t)}\,b(t)\,dt\,, \tag{35}$$

where $\Delta(t,x) \equiv \mathbf{I}^{\dagger}\mathbf{P}(t) \wedge \mathbf{y}(x)$. For a comparison to matrix methods, see [3] p. 235–237 or [4] p. 237–238.

4 Conclusion

If the reader is aware of both Geometric Algebra and Geometric Calculus, he or she may wonder why the title of this paper refers to the use of the former, but not the latter, even though clearly we are doing calculus in this paper. I'm not calling the content of this paper as being about Geometric Calculus because I consider it to be concern about differentiation with respect to a vector, not with respect to just a scalar, which is used in this paper.

[9] See Appendix 6 for more details.

The two results obtained in this paper are obtained conventionally by forming stacks of differential equations, which are then effectively organized into matrix equations and solved, or else subjected to determinants. It's all very straightforward, but, by comparison to the methods used herein, a tad messy. I've just made a cursory examination of the literature on the Variation of Parameters method and discovered no proofs of this theorem that do not use the linear algebra methods I just described. Therefore, the Geometric Algebra solution to it presented herein may be the only existing alternative to conventional proofs to the theorem.

The formulas derived here are the same as that derived by conventional methods, therefore the practical use of them would be no different than the use of their conventionally derived counterpart.

As a final comment, the constraint $\mathbf{v}' \cdot \mathbf{P} = 0$ provided us a valuable means to the algebraic solution to the Variation of Parameters. But we shouldn't stop there. $\mathbf{P}$ is a hyperplane generator and someone should investigate what's likely to be an important geometry concerning this space. Maybe you.

5 Appendix 0: The Wronskian in Linear Differential Equations

The use of the Wronskian in this paper is as a corollary to how it's used in a wider arena. In this paper, we wanted to know if a certain set of n functions of x, namely, the n solutions to the homogeneous differential equation (5), formed a linearly independent set

$$\{y_1, y_2, \ldots, y_n\} \tag{36}$$

Obviously, the above functions are intimately related to each other as co-solutions to some homogeneous differential equation.

But let's now assume we have a set of n arbitrarily chosen functions of x, none of them being identically zero,

$$\{g_1(x), g_2(x), \ldots, g_n(x)\}\,, \tag{37}$$

and we ask if there is a simple test to determine if together they form a linearly independent set of functions. Linear independence is a fundamental concept in linear algebra. So what does it mean?

Let S be a space of real-valued or complex-valued functions, as presented to us in (37). Let n scalars, as yet undetermined, be given as

$$\{c_1, c_2, \ldots, c_n\}\,. \tag{38}$$

Now, these functions are said to be linearly dependent if it is possible to solve for one of them in terms of the rest of them. So, let's say, without loss of generality, that we can solve for $g_1(x)$ in terms of the rest of them, as follows

$$g_1(x) = a_2 g_2(x) + a_3 g_3(x) + \ldots + a_n g_n(x)\,, \tag{39}$$

where the a_i's are constants, and in this case the g_i's form a linearly dependent set.

Now, if we put all the g's on the LHS, we can write (39) as

$$g_1(x) - a_2 g_2(x) - a_3 g_3(x) - \ldots - a_n g_n(x) = 0\,, \tag{40}$$

where the coefficient of the $g_1(x)$ term is unity. Let's instead put all these functions on a level playing field by rewriting (40) as

$$c_1 g_1(x) + c_2 g_2(x) + c_3 g_3(x) + \ldots + c_n g_n(x) = 0\,. \tag{41}$$

So, let's turn to cases. What if the only way to make the equality hold in (41) is for all the c_i's to be zero. Then no one of the $g_j(x)$'s can be solved in terms of the rest of them and we conclude that the functions are **not** linearly dependent. Another term for being 'not linearly dependent' is to be *linearly independent*.

Now, it's not possible for just one of the c_i's to be nonzero, say c_j, for then we would conclude that the functional value of $g_j(x)$ is zero, which we have assumed cannot be the case. Therefore, for a linear dependence to exist on the $g_i(x)$ of Eq. (41), there must be at least two nonzero coefficients of the c_i's.

From this point on, I will give examples in two dimensions to save ourselves from clutter. And we add one more condition to our functions so that we can use the so-called *Wronskian* analysis, and that conditions of that our functions of interest are all differentiable.

Let us now consider the linear dependence of two differentiable functions f and g. Then we form the equation

$$c_1 f(x) + c_2 g(x) = 0\,. \tag{42}$$

On taking the derivative of this equation, we get

$$c_1 f'(x) + c_2 g'(x) = 0\,. \tag{43}$$

Now, if we regard c_1 and c_2 as our two unknowns to solve for, then we can form the matrix equation out of these last two equations, to get

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,. \tag{44}$$

Let's abstract this matrix equation to

$$A\mathbf{c} = \mathbf{0}\,, \tag{45}$$

where

$$A = \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\,. \tag{46}$$

At this point, I'll invoke some results from linear algebra. First, if matrix A is invertible, then

$$\mathbf{c} = A^{-1}\mathbf{0} = \mathbf{0}\,, \tag{47}$$

which tells us that the only solutions for c_1 and c_2 are zeros, hence $f(x)$ and $g(x)$ are linearly independent of each other. And from linear algebra, we know that A is invertible if and only if $\det(A) = |\,A\,| \neq 0$.

Now, in the terminology of linear differential equations, the matrix we defined in Eq. (46) [we called the 'Wronski'] and its determinant is conventionally called 'Wronskian' and is usually written as W. Therefore, our conclusion is that the functions f and g are linearly independent if and only if their Wronskian is not zero. Otherwise, they are linearly dependent.

To generalize, when f_1, f_2, ..., and f_n represent n solutions to an nth-order linear homogeneous differential equation, the matrix A represents the Wronskian, and to ensure that the solutions are linearly independent, the Wronskian must be different than zero. Another reason we need the Wronskian to be nonzero is because it shows up in the denominator of our solution for the particular solution (35).

6 Appendix 1: Linear Differential Operators

This appendix quickly goes over the linearity of the differential operators discussed in this paper. I'll demonstrate on second-order differential equations, but the same is trivially true for nth-order differential equations.

First, what exactly is meant by a 'linear operator'? A linear operator distributes over addition and commutes with scalars. That is, if L is a linear operator over a space of objects, containing two arbitrary elements A and B of some space S, then

$$L(A + B) = L(A) + L(B)\,. \tag{48a}$$

Now, if α is a scalar such that αA is also in space S, then for L to be a linear operator, we also require that

$$L(\alpha A) = \alpha L(A)\,. \tag{48b}$$

Consider the differential equation

$$\frac{d^2}{dx^2}Y + a_1(x)\frac{d}{dx}Y + a_2(x)Y = b(x)\,. \tag{49}$$

Now, I'm claiming that the operator defined by

$$L \equiv \frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)\,, \tag{50}$$

so that (1) then takes the form

$$L(Y) = b(x)\,, \tag{51}$$

is an operator on the space of functions in the variable x.

Proof:

Let A and B be complex-valued functions. It's proven in calculus that the derivative by x distributes over A and B; hence,

$$\frac{d}{dx}(A+B) = \frac{d}{dx}(A) + \frac{d}{dx}(B)\,. \tag{52}$$

Therefore, multiplying by $a_1(x)$ will distribute as well:

$$a_1(x)\frac{d}{dx}(A+B) = a_1(x)\frac{d}{dx}(A) + a_1(x)\frac{d}{dx}(B)\,. \tag{53}$$

Similarly, it's clear that, on setting $d/dxA = A'$, then

$$\begin{aligned}\frac{d^2}{dx^2}(A+B) &= \frac{d}{dx}(A'+B') \\ &= \frac{d}{dx}(A') + \frac{d}{dx}(B') \quad \text{(using (52))} \\ &= \frac{d^2}{dx^2}(A) + \frac{d^2}{dx^2}(B)\,. \end{aligned} \tag{54}$$

And as for the zero-derivative function $a_2(x)$,

$$a_2(x)(A+B) = a_2(x)(A) + a_2(x)(B)\,. \tag{55}$$

Finally, we have that

$$\begin{aligned} L(Y_1+Y_2) &= \big[\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)\big](Y_1+Y_2) \\ &= \frac{d^2}{dx^2}(Y_1+Y_2) + a_1(x)\frac{d}{dx}(Y_1+Y_2) + a_2(x)(Y_1+Y_2) \\ &= \frac{d^2}{dx^2}(Y_1) + \frac{d^2}{dx^2}(Y_2) + a_1(x)\frac{d}{dx}(Y_1) + a_1(x)\frac{d}{dx}(Y_2) \\ &\qquad + a_2(x)(Y_1) + a_2(x)(Y_2) \\ &= \big[\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)\big](Y_1) + \big[\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)\big](Y_2) \\ &= L(Y_1) + L(Y_2) \end{aligned}$$

Done. And we can add any number of higher derivative terms to L and get the same result.

I leave to the reader to show that for some complex number c

$$L(cA) = cL(A)\,. \tag{56}$$

7 Appendix 2: Proof that $\mathbf{W}' = \mathbf{P} \wedge \mathbf{y}^{(n)}$

We begin with

$$\mathbf{P} \equiv \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)}\,, \tag{57}$$

where the primes denote differentiation by x. Now,

$$\mathbf{W} \equiv \mathbf{P} \wedge \mathbf{y}^{(n-1)} = \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \wedge \mathbf{y}^{(n-1)} \,. \tag{58}$$

Before we differentiate $\mathbf{W}$, let's look at a more generic case. Let

$$\mathbf{F} = \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \cdots \wedge \mathbf{Z} \,. \tag{59}$$

Then

$$\begin{aligned} \mathbf{F}' = \mathbf{A}' \wedge \mathbf{B} \wedge \mathbf{C} \wedge \cdots \wedge \mathbf{Z} + \mathbf{A} \wedge \mathbf{B}' \wedge \mathbf{C} \wedge \cdots \wedge \mathbf{Z} \\ + \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}' \wedge \cdots \wedge \mathbf{Z} + \cdots + \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \cdots \wedge \mathbf{Z}' \,. \end{aligned} \tag{60}$$

This is just the product rule for differentiation, where there is a term in the sum for each factor of the product, which gets differentiated. I won't prove here a vital fact: In any term made out of multiple wedge products, if any two factors of that product are equal, the whole product vanishes. So, if we differentiate over (58), we get

$$\begin{aligned} \mathbf{W}' = \mathbf{y}' \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \wedge \mathbf{y}^{(n-1)} + \mathbf{y} \wedge \mathbf{y}'' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \wedge \mathbf{y}^{(n-1)} \\ + \cdots + \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-1)} \wedge \mathbf{y}^{(n-1)} \\ + \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \wedge \mathbf{y}^{(n)} \,. \end{aligned} \tag{61}$$

So, the first term on the RHS vanishes because it has two factors of $\mathbf{y}'$, and the term after it vanishes because it has two factors of $\mathbf{y}''$, and this process of factors vanishing will continue until we reach the last factor, which has no manifestly identical factors in it, leaving us with

$$\mathbf{W}' = \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \wedge \mathbf{y}^{(n)} = \mathbf{P} \wedge \mathbf{y}^{(n)} \,. \tag{62}$$

8 Appendix 3: Proof that $W' + a_1(x)W = 0$

Because of (7), whatever $\mathbf{P}$ is

$$\langle \mathbf{P} L(\mathbf{y}) \rangle_n = 0 \,. \tag{63}$$

But, in particular, since

$$\mathbf{P} \equiv \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \,, \tag{64}$$

and $L(\mathbf{y})$ is given by

$$\begin{aligned} L(\mathbf{y}) &= \Big[\frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{d}{dx} + a_n(x) \Big] \, \mathbf{y} \\ &= \mathbf{y}^{(n)} + a_1(x) \mathbf{y}^{(n-1)} + \cdots + a_{n-1}(x) \mathbf{y}' + a_n(x) \, \mathbf{y} \,. \end{aligned} \tag{65}$$

Therefore,

$$\begin{aligned}\big\langle \mathbf{P}L(\mathbf{y})\big\rangle_n &= \big\langle \mathbf{P}\big[\,\mathbf{y}^{(n)} + a_1(x)\mathbf{y}^{(n-1)} + \cdots + a_{n-1}(x)\mathbf{y}' + a_n(x)\,\mathbf{y}\,\big]\big\rangle_n \\ &= \big\langle \big[\mathbf{y}\wedge\mathbf{y}'\wedge\cdots\wedge\mathbf{y}^{(n-2)}\big]\big[\,\mathbf{y}^{(n)} + a_1(x)\mathbf{y}^{(n-1)} + \cdots + a_n(x)\,\mathbf{y}\,\big]\big\rangle_n \,.\end{aligned} \tag{66}$$

Only two terms of this do not identically vanish, therefore

$$\begin{aligned}\big\langle \mathbf{P}L(\mathbf{y})\big\rangle_n &= \mathbf{y}\wedge\mathbf{y}'\wedge\cdots\wedge\mathbf{y}^{(n-2)}\wedge\mathbf{y}^{(n)} + a_1(x)\mathbf{y}\wedge\mathbf{y}'\wedge\cdots\wedge\mathbf{y}^{(n-2)}\mathbf{y}^{(n-1)} \\ &= \mathbf{W}' + a_1(x)\mathbf{W}\,.\end{aligned} \tag{67}$$

So, on combining this result with (63), we get that

$$\mathbf{W}' + a_1(x)\mathbf{W} = 0\,. \tag{68}$$

By multiplying this equation through on the right by $\mathbf{I}^\dagger$, we get the differential equation precursor to Abel's Identity:

$$W' + a_1(x)W = 0\,. \tag{69}$$

Integrating this, we have Abel's Identity:

$$W = c_1 \exp\left\{-\int a_1(x)dx\right\}\,. \tag{70}$$

9 Appendix 4: Proof that $\mathbf{W}_k + a_k(x)\mathbf{W} = 0$

We define the pseudoscalar quantity

$$\mathbf{W}_k \equiv \langle\, \mathbf{y}\mathbf{y}'\cdots\mathbf{y}^{(k-1)}\mathbf{y}^{(n)}\mathbf{y}^{(k+1)}\cdots\mathbf{y}^{(n-1)}\rangle_n\,, \tag{71}$$

and the scalar quantity

$$W_k \equiv \det(\mathbf{W}_k) = \mathbf{W}_k\mathbf{I}^\dagger\,. \tag{72}$$

We know that

$$L(\mathbf{y}) = \mathbf{y}^{(n)} + a_1(x)\mathbf{y}^{(n-1)} + \cdots + a_{n-1}(x)\mathbf{y}' + a_n(x)\,\mathbf{y} = 0\,. \tag{73}$$

Now, multiply $L(\mathbf{y}) = 0$ on the left by $\mathbf{y}\mathbf{y}'\cdots\mathbf{y}^{(k-1)}$ and on the right by $\mathbf{y}^{(k+1)}\cdots\mathbf{y}^{(n-2)}\mathbf{y}^{(n-1)}$ and take the pseudoscalar parts, the only surviving terms presenting

$$\begin{aligned}&\langle\, \mathbf{y}\mathbf{y}'\cdots\mathbf{y}^{(k-1)}\,[\,\mathbf{y}^{(n)}\,]\,\mathbf{y}^{(k+1)}\cdots\mathbf{y}^{(n-1)}\rangle_n + \\ &\qquad + \langle\, \mathbf{y}\mathbf{y}'\cdots\mathbf{y}^{(k-1)}\,[\,a_k(x)\mathbf{y}^{(k)}\,]\,\mathbf{y}^{(k+1)}\cdots\mathbf{y}^{(n-1)}\rangle_n = 0\,,\end{aligned} \tag{74}$$

to yield

$$\mathbf{W}_k + a_k(x)\mathbf{W} = 0\,. \tag{75}$$

10 Appendix 5: Proof that $\mathbf{v}' \cdot \mathbf{P} = 0$

We begin this proof with

$$\mathbf{P} = \mathbf{y} \wedge \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \,. \tag{76}$$

Then, if we define the m-blade $\mathbf{A}$ by

$$\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_m \,. \tag{77}$$

To demonstrate how this works, we present the general rule of taking the inner product of an arbitrary vector $\mathbf{b}$ and an arbitrary m-blade $\mathbf{A}$, then the rule to expand $\mathbf{b} \cdot \mathbf{A}$ is given by

$$\begin{aligned} \mathbf{b} \cdot \mathbf{A} = (\mathbf{b} \cdot \mathbf{a}_1)\, \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \cdots \wedge \mathbf{a}_m - (\mathbf{b} \cdot \mathbf{a}_2)\, \mathbf{a}_1 \wedge \mathbf{a}_3 \wedge \cdots \wedge \mathbf{a}_m \\ + \cdots + (-1)^{m+1} (\mathbf{b} \cdot \mathbf{a}_m)\, \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_{m-1} \,. \end{aligned} \tag{78}$$

Therefore,

$$\begin{aligned} \mathbf{v}' \cdot \mathbf{P} = (\mathbf{v}' \cdot \mathbf{y})\, \mathbf{y}' \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} - (\mathbf{v}' \cdot \mathbf{y}')\, \mathbf{y} \wedge \mathbf{y}'' \wedge \cdots \wedge \mathbf{y}^{(n-2)} \\ + \cdots + (-1)^{(n-2)+1} (\mathbf{v}' \cdot \mathbf{y}^{(n-2)})\, \mathbf{y} \wedge \mathbf{y}' \wedge \cdots \wedge \mathbf{y}^{(n-3)} \,. \end{aligned} \tag{79}$$

Now, the constraints we have chosen are

$$\langle\, \mathbf{v}'\mathbf{y}\,\rangle = \langle \mathbf{v}'\mathbf{y}' \rangle = \cdots = \langle\, \mathbf{v}'\mathbf{y}^{(n-2)} \rangle = 0 \,, \tag{80}$$

which can also be expressed as

$$\mathbf{v}' \cdot \mathbf{y} \;= \mathbf{v}' \cdot \mathbf{y}' = \cdots = \mathbf{v}' \cdot \mathbf{y}^{(n-2)} = 0 \,. \tag{81}$$

So, when we plug these values into (80), we get that all the terms on the RHS vanish. Therefore, we have shown that

$$\mathbf{v}' \cdot \mathbf{P} = 0 \,. \tag{82}$$

By the way, since $\mathbf{P}$ and $\mathbf{P}^{-1}$ differ only by a scalar multiple, namely, P^2, Eq. (82) lets us also claim that

$$\mathbf{v}' \cdot \mathbf{P}^{-1} = 0 \,, \tag{83}$$

which will be a useful result for later on.

11 Appendix 6: Arriving at $\langle \mathbf{v}'\mathbf{y}^{(n-1)} \rangle = b(x)$

To arrive at

$$\langle \mathbf{v}'\mathbf{y}^{(n-1)} \rangle = b(x) \,. \tag{84}$$

we need

$$L(\psi_0) = \langle L(\mathbf{v}\mathbf{y}) \rangle = b(x) \,. \tag{85}$$

To that end we set the constraints

$$\langle\, \mathbf{v}'\mathbf{y}\,\rangle = \langle \mathbf{v}'\mathbf{y}'\rangle = \langle \mathbf{v}'\mathbf{y}''\rangle = \cdots = \langle\, \mathbf{v}'\mathbf{y}^{(n-2)}\rangle = 0\,, \tag{86}$$

and from these we get the relation

$$\mathbf{v}' \cdot \mathbf{P} = 0\,. \tag{87}$$

By differentiating $\langle\, \mathbf{v}'\mathbf{y}\,\rangle = 0$ in (86) we get the other constraints we need. For example, we get

$$\langle\, \mathbf{v}''\mathbf{y} + \mathbf{v}'\mathbf{y}'\,\rangle \equiv 0\,. \tag{88}$$

However, from (29) we know that $\langle \mathbf{v}'\mathbf{y}'\rangle = 0$, therefore (88) becomes just $\langle \mathbf{v}''\mathbf{y}\rangle = 0$. If we continue to take derivatives of these constraints we find that all terms with $\mathbf{v}$ to first- or higher-order derivatives are zero except for the term $\langle \mathbf{v}'\mathbf{y}^{(n-1)}\rangle$. By expanding L in (28) all those terms with undifferentiated $\mathbf{v}$ add up to

$$\langle \mathbf{v}L(\mathbf{y})\rangle = 0 \tag{89}$$

since $L(\mathbf{y}))\rangle = 0$. This leaves the only surviving term to be $\langle\, \mathbf{v}'\mathbf{y}^{(n-1)}\rangle$. Thus (94) reduces to (88).

Let's put in a few convincing steps: I will show how to move from (86) to

$$\langle\, \mathbf{v}''\mathbf{y}\,\rangle = \langle \mathbf{v}''\mathbf{y}'\rangle = \cdots = \langle\, \mathbf{v}''\mathbf{y}^{(n-2)}\rangle = 0\,, \tag{90}$$

and so on.

By differentiating $\langle\, \mathbf{v}'\mathbf{y}'\,\rangle = 0$ in (86) we get the another constraint we need:

$$\langle\, \mathbf{v}''\mathbf{y}' + \mathbf{v}'\mathbf{y}''\,\rangle \equiv 0\,, \tag{91}$$

but $\langle\, \mathbf{v}'\mathbf{y}''\,\rangle = 0$ therefore,

$$\langle\, \mathbf{v}''\mathbf{y}'\,\rangle \equiv 0\,, \tag{92}$$

and so on. The next row of derivatives is

$$\langle\, \mathbf{v}'''\mathbf{y}\,\rangle = \langle \mathbf{v}'''\mathbf{y}'\rangle = \cdots = \langle\, \mathbf{v}'''\mathbf{y}^{(n-3)}\rangle = 0\,, \tag{93}$$

where we stop at $\mathbf{v}'''\mathbf{y}^{(n-3)}$ because the largest derivative operator is D_x^n.

Now let's put the pieces together. Equation (28) becomes

$$\langle L(\mathbf{vy})\rangle = \langle [\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)](\mathbf{vy})\rangle = b(x)\,. \tag{94}$$

And this distributes to be

$$\Big\langle \frac{d^n}{dx^n}(\mathbf{vy})\Big\rangle + a_1(x)\Big\langle \frac{d^{n-1}}{dx^{n-1}}(\mathbf{vy})\Big\rangle + \cdots + a_{n-1}(x)\Big\langle \frac{d}{dx}(\mathbf{vy})\Big\rangle + a_n(x)\Big\langle(\mathbf{vy})\Big\rangle = b(x)\,. \tag{95}$$

Let's examine the leading term, which distributes to be

$$
\begin{aligned}
\Big\langle \frac{d^n}{dx^n}(\mathbf{v}\mathbf{y}) \Big\rangle &= \Big\langle \frac{d^{n-1}}{dx^{n-1}}(\cancelto{0}{\mathbf{v}'\mathbf{y}} + \mathbf{v}\mathbf{y}') \Big\rangle \\
&= \Big\langle \frac{d^{n-2}}{dx^{n-2}}(\cancelto{0}{\mathbf{v}'\mathbf{y}'} + \mathbf{v}\mathbf{y}'') \Big\rangle \\
&\quad\vdots \\
&= \Big\langle \frac{d}{dx}(\mathbf{v}\mathbf{y}^{(n-1)}) \Big\rangle \\
&= \Big\langle \mathbf{v}'\mathbf{y}^{(n-1)} \Big\rangle + \Big\langle \mathbf{v}\mathbf{y}^{(n)}) \Big\rangle \, .
\end{aligned} \tag{96}
$$

Therefore,

$$
\begin{aligned}
\langle L(\mathbf{v}\mathbf{y}) \rangle = \Big\langle \mathbf{v}'\mathbf{y}^{(n-1)} \Big\rangle &+ \Big\langle \mathbf{v}\mathbf{y}^{(n)} \Big\rangle \\
&+ a_1(x)\Big\langle \text{singular terms} + \mathbf{v}\mathbf{y}^{(n-1)} \Big\rangle \\
&+ a_2(x)\Big\langle \text{singular terms} + \mathbf{v}\mathbf{y}^{(n-2)} \Big\rangle \\
&+ \cdots + \\
&+ a_{n-1}(x)\Big\langle \text{singular terms} + \mathbf{v}\mathbf{y}' \Big\rangle \\
&+ a_n \Big\langle \mathbf{v}\mathbf{y} \Big\rangle \\
= b(x) \, .
\end{aligned} \tag{97}
$$

Now, if we sum up all the terms in which $\mathbf{v}$ is not differentiated, we get $\langle \mathbf{v}L(\mathbf{y}) \rangle$ which is zero because $L(\mathbf{y}) = 0$. Finally, if we ignore all the singular terms (their values are zero), we end up with

$$
\Big\langle \mathbf{v}'\mathbf{y}^{(n-1)} \Big\rangle = b(x) \, . \tag{98}
$$

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