Understanding the Mystery of Clifford Algebra

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Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, July, 1995. Some scientists seem to believe that the unreasonable effectiveness of Clifford algebras in physics is too good to be true. They resist using it or allowing themselves a chance to understand it because of psychological and/or philosophical prejudices. Some of them even admit all this. This short essay attempts to alleviate those doubts.

1 Introduction

In 1991, the well-known scientist, E. T. Jaynes, wrote in the advanced physics book, *The Electron* [1, p. 5] a paper called "Scattering of Light by Free Electrons," the following to complain of David Hestenes's use of Clifford algebra in his SpaceTime Algebra [2]:

> Physicists go into a state of mental shock when they see a single equation which purports to represent the sum of a scalar and a vector. All of our training, from childhood on, has ground into us that one must never dream of doing such an absurd thing. . . .

What bothers Jaynes is Hestenes's use of equations such as

$$uv = u \cdot v + u \wedge v \,. \tag{1}$$

where $u \cdot v$ is a scalar and $u \wedge v$ is a "vector" (really a bivector, unless interpreted projectively). Jaynes tries to make sense of this by suggesting that the '+' and '=' signs in (1) have different meanings than in ordinary scalar or vector algebra. He suggests that it's like adding apples and oranges together.

Well, in one sense Jaynes is right but in another he is wrong. If interpreted broadly enough the '+' and '=' signs can be understood to have the same meaning for scalar, vector, and multivector algebra, where a "multivector" is an arbitrary element of a Clifford algebra. We will define addition as the process of adding "like" things together.

Now, two things are said to be "alike" if the both belong to the same mathematical object, such as a group, a ring, a field, a vector space, or a general algebra. Each of these objects have in common 1) at least one binary (possibly n-ary) operator that associates for any two (or n) elements of a given set another element of the set, and 2) all have a way of defining the equality or inequality of any two elements of the set, and 3) for our purposes here, we take the binary (n-ary) operations to be associative. Both groups and vector spaces allow for only one binary operation defined for elements of the sets they're defined on, though the vector space also allows for the multiplication of the vectors by scalars. It takes some but not much faith to accept these latter two mathematical objects because each is well within the grasp of our intuition.

But this is not the case with more complicated objects, such as an algebra, of which a ring and a field are special types. A generally accepted definition of "algebra" is lacking in the literature, and I've no doubt that this contributes to the lack of faith in general algebras being applied to physics. The most general way to define an algebra is to start with a *module*, a mathematical object that allows the multiplication of one kind of mathematical object by another, such as the multiplication of a vector by a supposedly different thing, a scalar, which is our familiar notion of a vector space. For our purposes here, however, let's just assume that an algebra is any set of elements which are closed under addition '+' of elements and multiplication of elements of a *given* set. Examples are the ring of integers, the field of reals or complex numbers, and matrix algebras.

But we must ask ourselves on what basis we are going to label things like scalars and vectors as "different." Is it on the basis of our intuitions or is it on the basis of mathematical concinnity. The history of mathematics shows that when people restrict themselves to merely intuitive justifications then they lose the great potential of mathematics. At one time it was considered impossible to justify adding positive and negative numbers, remember? For the sake of argument, let's adopt an abstract and pragmatic notion of "likeness" of numbers. Let's say that there are generally two forms of "alikeness" to be defined on the elements of an algebra. The first is called "additive alikeness" and it applies to any two elements of any algebra simply because any two elements of an algebra can be added together in a meaningful way, by which I mean that their sum is still an element of the algebra. You see, in the abstract world of algebras, the only meaning that can be assigned to any mathematical object in creation is how it relates to the algebra formally. The first way to relate is either to be in the set or not to be in the set.

The next way that any two elements in a set can be related is by another abstract assumption or definition placed on the set. One way to go is to define equivalency on the set, but this is not the direction to go to best demonstrate the quandary over Clifford algebras. That direction is in the notion of "selection." But before I can properly define selection I must first explain equality '=' on an algebra. When we see an equation such as

$$A = B\,, \tag{2}$$

we know it to mean that everything taken together on the left-hand side of the '=' is equal to everything taken together on the right-hand side of the '='. I have not attempted to present this notion rigorously because this aspect of an algebra is not a point of contention. The contentions in the case of Clifford algebra (or at least the case of *geometric algebra* in which the scalars are taken to be the real numbers) is that on top of the axioms of addition and multiplication of multivectors is also the axiom of *grade*, or *step*, selection. This axiom is as follows: Given Equation (2), then for each k for $0 \le k \le n$

$$\langle A\rangle_k \;=\; \langle B\rangle_k\,, \tag{3}$$

where the notation $\langle\rangle_k$ represents the kth-graded part of the multivector it operates on.

Clifford algebra is really a sophisticated way to encode, manipulate, and retrieve information. And geometric algebra is a sophisticated way to encode and retrieve geometric information. I cannot explain just how this magical process works so well for the applications to geometry and much less for physics, for I see it as part of the mysterious "unreasonable effectiveness of mathematics." However, I can credit it to the advantages of having scalars, vectors, and bivector as additively alike, by intentionally building a geometric interpretation into the algebra (this is done by choosing axioms to facilitate a geometrical interpretation), by adopting an associative algebra from the start, and by much more.

But even though I can't explain the magic of adding 'apples and oranges,' I can demonstrate that we've been doing it in mathematics from childhood! We add reals and fractions as though they are the same, though they have different axiomatic foundations. We add primes and composites as though they are the same, though they differ in important ways. We add reals and imaginaries together to get "complex" numbers, and we even define a selection rule on them. For example, if the equation in (2) represents complex numbers, then we write

$$\langle A\rangle_{\mathrm{Re}} = \;\langle B\rangle_{\mathrm{Re}}\,, \tag{4a}$$

$$\langle A\rangle_{\mathrm{Im}} = \;\langle B\rangle_{\mathrm{Im}}\,. \tag{4b}$$

In chemistry we have no hesitation to write

$$x_1\,\mathrm{HNO_3} = x_2\,\mathrm{NO_2} + x_3\,\mathrm{O_2} + x_4\,\mathrm{H_2O}\,, \tag{5}$$

and then to decode part of this encoding of information as

$$\langle x_1\,\mathrm{HNO_3}\rangle_{\mathrm{O}} \;=\; \langle x_2\,\mathrm{NO_2} + x_3\,\mathrm{O_2} + x_4\,\mathrm{H_2O}\rangle_{\mathrm{O}} \tag{6}$$

to represent the conservation of oxygen to get

$$\mathrm{O}: \qquad 3x_1 = 2x_2 + 2x_3 + x_4\,. \tag{7}$$

In a word problem in the last article we thought of ratiator fluid as a mixture of water (W) and alcohol (A) to get

$$W_1 \oplus A_1 + \Delta W = W_2 \oplus A_2\,, \tag{8a}$$

or

$$(W_1 + \Delta W) \oplus A_1 = W_2 \oplus A_2 \,. \tag{8b}$$

Indeed, on the surface this looks like two different additions, but they're not really. If we drop the circles on the plus signs in (8a) we can recapture the essence of (8b) by introducing selection on (8a), giving us for the case of water

$$\langle W_1 + A_1 + \Delta W \rangle_W = \langle W_2 + A_2 \rangle_W \tag{9a}$$

or

$$W_1 + \Delta W = W_2 \,. \tag{9b}$$

The concept of selection is one of the most powerful concepts for solving problems and for creating identities, for by it we can virtually emplace into a selector any object of the algebra we like which the selector maps to zero. If the concept of selection were taught to students from the beginning of their education then people would have no difficulty with adding apples and oranges, which, by the way, can be added as follows:

$$A_1 \oplus O_1 + A_2 \oplus O_2 = (A_1 + A_2) \oplus (O_1 + O_2) \,. \tag{10}$$