Finding the Curve that Focuses Parallel Rays to a Single Point

P. Reany

June 27, 2022

Abstract

This paper uses Geometric Algebra to solve for the curve that focuses parallel rays to a single point. The curve will be shown to be a parabola and the point its focus. A basic knowledge of geometric algebra and first-year calculus is assumed.

1 Introduction

Consider the limiting case of an infinitely shallow 2-D reflecting structure (curve) in $\mathbb{R}^2$. It will reflect all light rays that hit it by the rule *the angle of reflection is equal to the angle of incidence*. We demand that this reflecting structure sends all reflecting rays to the point $\mathbf{Q} = (0, w)$ with $w > 0$. We also demand that this reflecting structure pass through the point $(0, 0)$. Lastly, we'll assume that the equation of this curve is at most quadratic. The situation is depicted in the figure below.

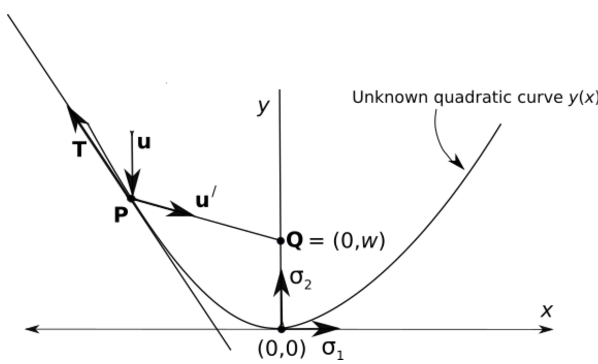

Figure 1. Sure, our so-called unknown curve is a parabola, but we have to prove that and solve for its coefficients. The formula for $\mathbf{u}'$ is given by $-\mathbf{T}(-\mathbf{u})\mathbf{T}$, where $\mathbf{u} = -\boldsymbol{\sigma}_2$ is a unit vector, and where $\mathbf{T}$ is a tangent vector to the curve at the point $\mathbf{P}$. A priori, it's possible that there is no quadratic curve that suits the given requirements. Well, we'll see.

2 Solution

For starters, we're assuming the curve to be a polynomial, so it's differentiable everywhere. Therefore, it has a well-defined tangent line to every point $\mathbf{P}$ on the curve. By the way, since the point $\mathbf{P}$ is arbitrarily chosen, what we prove about the reflection at this point is true for every point on the curve. As for the curve, we're assuming it has the polynomial form:

$$f(x) = ax^2 + bx\,, \tag{1}$$

where the constant term is zero to satisfy the requirement that the curve goes through the origin. Anyway, the slope function $f'(x)$ is given by

$$f'(x) = 2ax + b\,. \tag{2}$$

The geometric algebra encoding for a reflection about a vector $\mathbf{T}$ is given by

$$\mathbf{u}' = -\hat{\mathbf{T}}(-\mathbf{u})\hat{\mathbf{T}}\,. \tag{3}$$

By requirement in our problem, $\mathbf{u}'$ must be a unit vector along the direction $\pm(\mathbf{Q}-\mathbf{P})$, and it won't matter which sign we choose. Also, by construction, $\mathbf{u} = -\boldsymbol{\sigma}_2$. The last constraint we need to satisfy is

$$[\,\hat{\mathbf{T}}(\boldsymbol{\sigma}_2)\hat{\mathbf{T}}\,] \wedge (\mathbf{P}-\mathbf{Q}) = 0\,. \tag{4}$$

Now, since the expression on the LHS is being set equal to zero, we don't have to worry about normalizing the tangent vector, hence, we can simplify (4) down to

$$[\,\mathbf{T}\boldsymbol{\sigma}_2\,\mathbf{T}\,] \wedge (\mathbf{P}-\mathbf{Q}) = 0\,. \tag{5}$$

So, now we need a tangent vector $\mathbf{T}$.

$$\mathbf{T} = (\Delta x, \Delta y) = \Delta x(1, \frac{\Delta y}{\Delta x}) = \Delta x(1, f'(x))\,. \tag{6}$$

We'll drop the nonzero factor of Δx and, with $y' = f'(x)$, write

$$\mathbf{T} = (1, y') = \boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2\,. \tag{7}$$

We'll get the constraints we need on a and b in (1) by reducing Eq. (5) to a sum of terms that are each scalar multiples of the bivector $\boldsymbol{\sigma}_{12} = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2$.

So, now we calculate $\mathbf{T}\boldsymbol{\sigma}_2\,\mathbf{T}$.

$$\begin{aligned}
\mathbf{T}\boldsymbol{\sigma}_2\,\mathbf{T} &= (\boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2) \\
&= (\boldsymbol{\sigma}_{12} + y')(\boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2) \\
&= -\boldsymbol{\sigma}_2 + 2y'\boldsymbol{\sigma}_1 + (y')^2\boldsymbol{\sigma}_2 \\
&= 2y'\boldsymbol{\sigma}_1 + [\,(y')^2 - 1\,]\boldsymbol{\sigma}_2
\end{aligned} \tag{8}$$

What about $\mathbf{P} - \mathbf{Q}$?

$$\mathbf{P} - \mathbf{Q} = (x, y) - (0, w) = x\boldsymbol{\sigma}_1 + (y - w)\boldsymbol{\sigma}_2 \,. \tag{9}$$

Now we're ready for (5):

$$\begin{aligned}
[\,\mathbf{T}\,\mathbf{u}\,\mathbf{T}\,] \wedge (\mathbf{P} - \mathbf{Q}) &= \big[\,2y'\boldsymbol{\sigma}_1 + [\,(y')^2 - 1\,]\boldsymbol{\sigma}_2\,\big] \wedge [\,x\boldsymbol{\sigma}_1 + (y - w)\boldsymbol{\sigma}_2\,] \\
&= 2y'(y - w)\boldsymbol{\sigma}_{12} - [\,(y')^2 - 1\,]x\,\boldsymbol{\sigma}_{12} \\
&= \{2y'(y - w) - [\,(y')^2 - 1\,]x\}\boldsymbol{\sigma}_{12} \\
&= 0\,.
\end{aligned} \tag{10}$$

So, if this procedure is to work, the constraints we need on a and b will be evident when we expand

$$2y'(y - w) - [\,(y')^2 - 1\,]x = 0\,, \tag{11}$$

and then compare coefficients to (1). Substituting in from (1) and (2), we get, after much simplification,

$$2abx^2 + (-4aw + b^2 + 1)x - 2bw = 0\,. \tag{12}$$

First, on comparing this to (1), we see that $-2bw = 0$, and since $w \neq 0$, then $b = 0$, leaving us with

$$(-4aw + 1)x = 0\,. \tag{13}$$

On setting this last coefficient to zero, we get

$$w = \frac{1}{4a}\,, \tag{14}$$

which is the expected result, although we need to solve for a in this case, to get for our quadratic polynomial

$$f(x) = \frac{1}{4w}x^2\,, \tag{15}$$

which, in standard form, would be written as

$$4w(y + k) = (x + h)^2\,, \tag{16}$$

with $h = k = 0$, since the vertex is at point $(0, 0)$.

3 Conclusion

Instead of using geometric algebra to help in proving this theorem, I could have used the Gibbs's inner product and cross product, and have accomplished the task. Either way, let's not forget the point of the theorem: There exists a quadratic solution to the existence problem of whether or not incoming parallel rays can be focused to the single point by reflecting them off a surface defined by a quadratic polynomial.