Theorem on Reversing an Inner Product

P. Reany

May 17, 2022

1 The Theorem

The following is a proof of one of the fundamental results on inner products.

Theorem: Given that $A = \langle A \rangle_r$ and $B = \langle B \rangle_s$, then

$$A \cdot B = (-1)^{s(r-s)} B \cdot A \qquad \text{for } r \geq s\,. \tag{1}$$

The first thing I want to point out is that one could modify the coefficient on the RHS of (1) a lot, without changing its value. For example, let n be any integer, then

$$(-1)^n = (-1)^{-n}\,. \tag{2}$$

Now, let n and m be any integers, then

$$(-1)^n = (-1)^{n \pm 2m}\,. \tag{3}$$

Or, put another way, for integers k and ℓ,

$$(-1)^k = (-1)^\ell \tag{4}$$

any time k and ℓ differ by an even number. Lastly,

$$(-1)^{t^2 - t} = (-1)^{t(t-1)} = 1 \tag{5}$$

for all integer values of t, because whatever t is, either t or $t-1$ will be even.

For example, we could multiply by $(-1)^{2s^2}$ on the RHS of (1), to get the equivalent form,

$$A \cdot B = (-1)^{s(r+s)} B \cdot A \qquad \text{for } r \geq s\,. \tag{6}$$

2 Three Lemmas

Lemma 1:

For k a positive integer,

$$A_k^\dagger = (-1)^{k(k-1)/2} A_k\,. \tag{7}$$

Lemma 2:

For any multivector M,

$$(M^\dagger)^\dagger = M\,. \tag{8}$$

Lemma 3:

For any multivectors M and N,

$$\langle MN \rangle_\ell^\dagger = \langle N^\dagger M^\dagger \rangle_\ell \,. \tag{9}$$

Corollary:

If M is an s-vector and N is an r-vector, then, using (7)

$$\langle MN \rangle_\ell^\dagger = (-1)^{r(r-1)/2}(-1)^{s(s-1)/2}\langle NM \rangle_\ell \,. \tag{10}$$

3 Proof:

Set $\ell = r - s$, then

$$\begin{aligned} A \cdot B &= \langle AB \rangle_\ell \\ &= (\langle AB \rangle_\ell^\dagger)^\dagger \\ &= (\langle B^\dagger A^\dagger \rangle_\ell)^\dagger \\ &= (-1)^{r(r-1)/2}(-1)^{s(s-1)/2}(\langle BA \rangle_\ell)^\dagger \\ &= (-1)^{r(r-1)/2}(-1)^{s(s-1)/2}(-1)^{\ell(\ell-1)/2}\langle BA \rangle_\ell \\ &= (-1)^{s(r-s)} B \cdot A \,. \end{aligned} \tag{11}$$

The busy-work part of the proof follows. We start by ignoring the factors of one-half for the time being.

$$\begin{aligned} r(r-1)/2 + s(s-1) + \ell(\ell-1) &= r(r-1) + s(s-1) + (r-s)(r-s-1) \\ &= r^2 - r + s^2 - s + r^2 - 2rs + s^2 - r + s \\ &= 2r^2 - 2r + 2s^2 - 2rs \,. \end{aligned}$$

Now we divide by 2 and exponentiate:

$$(-1)^{r^2 - r + s^2 - rs} = (-1)^{s^2 - rs} = (-1)^{s(s-r)} = (-1)^{s(r-s)} \,. \tag{12}$$