A solved problem in rotational dynamics

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Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, July 1992. What's special about the solution here is that it uses geometric algebra.[1]

Given an axially symmetric rigid body rotating about a fixed point under no torques, and given that the angle between the symmetry axis and the rotational velocity vector $\boldsymbol{\omega}$ is α, show that the tangent of the angle γ between $\boldsymbol{\omega}$ and the angular momentum vector $\mathbf{L}$ is given by

$$\tan\gamma = \frac{(I_s - I)\tan\alpha}{I_s + I\tan^2\alpha} \tag{1}$$

where I_s is the moment of inertia about the symmetry axis and I is the moment of inertia about the axis normal the symmetry axis, which is the z-axis.

SOLUTION:

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal frame along principal directions of the body, where $\mathbf{e}_3$ is along the symmetry axis. Let θ be the angle between the symmetry axis and $\mathbf{L}$, where

$$\mathbf{L} = I\omega_x\mathbf{e}_1 + I\omega_y\mathbf{e}_2 + I_s\omega_z\mathbf{e}_3 \tag{2}$$

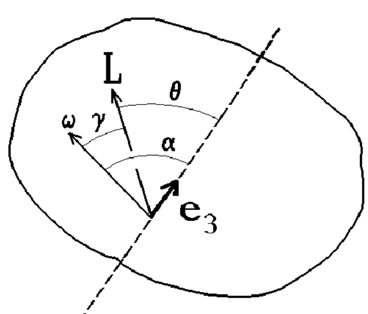

Figure 1. The relevant vectors and angles displayed. The dashed line is the axis of symmetry. Vectors $\mathbf{L}$, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ are coplanar.

[1]For references to geometric algebra, see the bibiography at the end of the article.

The following results will be useful:

$$\tan\alpha = \frac{|\boldsymbol{\omega}\wedge\mathbf{e}_3|}{\boldsymbol{\omega}\cdot\mathbf{e}_3} = \frac{\sqrt{\omega_x^2+\omega_y^2}}{\omega_z} \tag{3}$$

$$\tan\theta = \frac{|\mathbf{L}\wedge\mathbf{e}_3|}{\mathbf{L}\cdot\mathbf{e}_3} = \frac{\sqrt{L_x^2+L_y^2}}{L_z} = \frac{I}{I_s}\tan\alpha \tag{4}$$

$$\tan\gamma = \frac{|\mathbf{L}\wedge\boldsymbol{\omega}|}{\mathbf{L}\cdot\boldsymbol{\omega}} \tag{5}$$

Thus we can solve for $\tan\gamma$ in terms of $\mathbf{a}$ if we can solve for $\mathbf{L}\wedge\omega$ and $\mathbf{L}\cdot\omega$ in terms of α. The following step is key—a virtual emplacement based on the fact that $\mathbf{e}_3^2 = 1$ and that the geometric product is associative.

$$\mathbf{L}\boldsymbol{\omega} = \mathbf{L}(\mathbf{e}_3\mathbf{e}_3)\boldsymbol{\omega} = (\mathbf{L}\mathbf{e}_3)(\mathbf{e}_3\boldsymbol{\omega}) \tag{6}$$

$$\mathbf{L}\cdot\boldsymbol{\omega} + \mathbf{L}\wedge\boldsymbol{\omega} = (\mathbf{L}\cdot\mathbf{e}_3 + \mathbf{L}\wedge\mathbf{e}_3)(\mathbf{e}_3\cdot\boldsymbol{\omega} + \mathbf{e}_3\wedge\boldsymbol{\omega}) \tag{7}$$

On equating scalar and bivector parts, respectively, we get

$$\mathbf{L}\cdot\boldsymbol{\omega} = \mathbf{L}\cdot\mathbf{e}_3\mathbf{e}_3\cdot\boldsymbol{\omega} + |\mathbf{L}\wedge\mathbf{e}_3||\boldsymbol{\omega}\wedge\mathbf{e}_3| = L_z\omega_z + |\mathbf{L}\wedge\mathbf{e}_3||\boldsymbol{\omega}\wedge\mathbf{e}_3|\,, \tag{8}$$

$$\mathbf{L}\wedge\boldsymbol{\omega} = \mathbf{L}\cdot\mathbf{e}_3\mathbf{e}_3\wedge\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\mathbf{e}_3\mathbf{L}\wedge\mathbf{e}_3 = L_z\mathbf{e}_3\wedge\boldsymbol{\omega} - \omega_z\mathbf{L}\wedge\mathbf{e}_3\,. \tag{9}$$

Since all the bivectors of the last equation are coplanar, we can easily scalarize it, we get

$$|\mathbf{L}\wedge\boldsymbol{\omega}| = \mathbf{L}\cdot\mathbf{e}_3|\mathbf{e}_3\wedge\boldsymbol{\omega}| - |\boldsymbol{\omega}\cdot\mathbf{e}_3||\mathbf{L}\wedge\mathbf{e}_3| = L_z|\mathbf{e}_3\wedge\boldsymbol{\omega}| - \omega_z|\mathbf{L}\wedge\mathbf{e}_3|\,. \tag{10}$$

So that

$$\begin{aligned}\tan\gamma &= \frac{L_z|\mathbf{e}_3\wedge\boldsymbol{\omega}| - \omega_z|\mathbf{L}\wedge\mathbf{e}_3|}{L_z\omega_z + |\mathbf{L}\wedge\mathbf{e}_3||\boldsymbol{\omega}\wedge\mathbf{e}_3|} \\ &= \frac{|\mathbf{e}_3\wedge\boldsymbol{\omega}|/\omega_z - |\mathbf{L}\wedge\mathbf{e}_3|/L_z}{1 + \dfrac{|\mathbf{L}\wedge\mathbf{e}_3|}{L_z}\dfrac{|\boldsymbol{\omega}\wedge\mathbf{e}_3|}{\omega_z}} \\ &= \frac{(I_s - I)\tan\alpha}{I_s + I\tan^2\alpha}\end{aligned} \tag{11}$$

And this concludes the proof.

David Hestenes and Garret Sobczyk. 1984. *Clifford Algebra to Geometric Calculus*. D. Reidel Publishing Co. Dordrecht, Holland.
David Hestenes. 1986. *New Foundations for Classical Mechanics*. D. Reidel Publishing Co. Dordrecht, Holland.