Formula for the Location of the Intersection of Two Tangents of a Circle

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Abstract

I recently saw this problem solved by Michael Penn on his YouTube channel, from March 16, 2022, “A nice geometry problem with complex numbers.” Penn solved the problem using complex numbers. My goal here is to solve the problem using vectors and then the use geometric algebra to translate between Penn's solution and my own vector solution.

Introduction

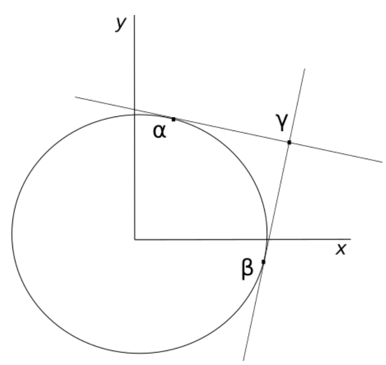

Figure 1. The two tangent lines meet on the circle at points α and β, the lines intersecting at point γ. Determine γ as a function of α and β. This is a unit circle.

Given the graphic in Fig. 1, find a formula for the location of point γ in the complex plane as Michael Penn presented the solution of the location of point γ in the complex plane as

$$\gamma = \frac{2\alpha\beta}{\alpha+\beta}\,, \tag{1}$$

where Penn treats this circle as having radius unity.

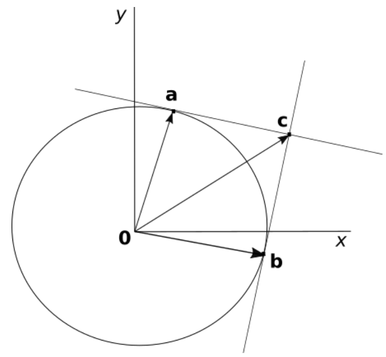

Figure 2. The two tangent lines meet on the circle at points **a** and **b**, the lines intersecting at point **c**. Determine **c** as a function of **a** and **b**. This circle has radius R.

Solution using vectors

We know that the vector **a** is perpendicular to the line $\mathbf{c}-\mathbf{a}$ at the point **a**. From this we get that

$$\mathbf{a}\cdot(\mathbf{c}-\mathbf{a}) = 0\,, \tag{2}$$

from which we have that

$$\mathbf{a}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2\,, \tag{3}$$

which will soon come in handy. We likewise know that the the vector **b** is perpendicular to the line $\mathbf{c}-\mathbf{b}$ at the point **b**. And given that

$$|\,\mathbf{a}\,| = |\,\mathbf{b}\,| = R\,, \tag{4}$$

it's easy to show by the Pythagorean Theorem that triangles $\triangle\mathbf{a0c}$ and $\triangle\mathbf{b0c}$ are congruent. Hence, identifying corresponding angles, we get

$$\angle\mathbf{a0c} = \angle\mathbf{b0c}\,. \tag{5}$$

Thus, if we think of **a** and **b** as basis vectors, **c** must by symmetry be representable as equal parts of both of them. Thus,

$$\mathbf{c} = \lambda(\mathbf{a}+\mathbf{b})\,, \tag{6}$$

where λ is a scalar, that is, a real number, which now we'll solve for. Dotting (6) through by **a**, we get

$$\mathbf{a}\cdot\mathbf{c} = \lambda(\mathbf{a}^2+\mathbf{a}\cdot\mathbf{b})\,. \tag{7}$$

Using the result from (3), we get that

$$\mathbf{a}^2 = \lambda(\mathbf{a}^2+\mathbf{a}\cdot\mathbf{b})\,. \tag{8}$$

On solving this for λ, we get

$$\lambda = \frac{\mathbf{a}^2}{\mathbf{a}^2 + \mathbf{a}\cdot\mathbf{b}} = \frac{1}{1 + \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}}\,. \tag{9}$$

On substituting this into (6), we get that

$$\mathbf{c} = \frac{\mathbf{a}+\mathbf{b}}{1 + \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}}\,. \tag{10}$$

So, after we've developed the transformation rules in geometric algebra to convert between complex numbers and vectors, we can show that (1) is equivalent to (10).

Complex numbers in geometric algebra

Let $\mathcal{G}_2$ be the geometric algebra of the plane of the linear combinations of basis vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, where $\boldsymbol{\sigma}_1$ is the unit vector along the x-axis and $\boldsymbol{\sigma}_2$ is the unit vector along the y-axis. The even subalgebra of $\mathcal{G}_2$, $\mathcal{G}_2^+$, is the algebra formed by the scalar and bivector (pseudoscalar) elements of $\mathcal{G}_2$. The set of all vectors of $\mathcal{G}_2$ do not form a subalgebra of $\mathcal{G}_2$. Since the even elements of $\mathcal{G}_2$ commute with each other, $\mathcal{G}_2^+$ is isomorphic to the complex numbers.

The unit imaginary of $\mathcal{G}_2^+$ is given nonuniquely by

$$\mathbf{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\,, \tag{11}$$

where

$$\mathbf{i}^2 = -1\,. \tag{12}$$

Let $\mathbf{x}$ be any vector in the plane, then

$$\mathbf{x}\mathbf{i} = -\mathbf{i}\mathbf{x}\,. \tag{13}$$

Now, we'll see how to map vectors into complex numbers. Let $\mathbf{x}$ be any vector in the plane, given by

$$\mathbf{x} = x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2\,, \tag{14}$$

in the usual way. Next, we multiply through on the left by $\boldsymbol{\sigma}_1$ to get

$$\boldsymbol{\sigma}_1\mathbf{x} = x + y\mathbf{i} = z\,, \tag{15}$$

where we used (11). If, instead, we have the complex number z and we wish to find its corresponding vector $\mathbf{x}$, we just multiply (15) through by $\boldsymbol{\sigma}_1$ on the left, to get

$$\mathbf{x} = \boldsymbol{\sigma}_1 z\,. \tag{16}$$

On using the reversion operator across (15) we have

$$z^\dagger = \mathbf{x}\boldsymbol{\sigma}_1 = x - y\mathbf{i}\,. \tag{17}$$

Hence, the reversion operation on our "complex" numbers fulfills the role of complex conjugation.

Finally, on taking the reversion of (16), we get that

$$\mathbf{x} = z^{\dagger}\boldsymbol{\sigma}_1 \,. \tag{18}$$

Thus, from this last equation and (16)

$$\boldsymbol{\sigma}_1 z = z^{\dagger}\boldsymbol{\sigma}_1 \,. \tag{19}$$

Therefore, on multiplying through by $\boldsymbol{\sigma}_1$ on the right, we have that

$$\boldsymbol{\sigma}_1 z\, \boldsymbol{\sigma}_1 = z^{\dagger} \,. \tag{20}$$

Now, some useful lemmas:

Lemma 1

The product of two vectors is an even element and thus is will commute with all other even elements (complex numbers). Thus, if $\mathbf{p}$ and $\mathbf{q}$ are any two vectors in $\mathcal{G}_2$ and ω is any element of $\mathcal{G}_2^+$ then

$$\mathbf{pq}\,\omega = \omega\,\mathbf{pq} \,. \tag{21}$$

Lemma 2

Given $\mathbf{a}$, $\mathbf{b}$, and R as defined above

$$(\mathbf{a}+\mathbf{b})^2 = 2R^2(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) \,. \tag{22}$$

Proof:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})^2 &= \mathbf{a}^2 + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2 \\ &= R^2 + 2R^2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} + R^2 \\ &= 2R^2(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) \,. \end{aligned} \tag{23}$$

Lemma 3

Given $\mathbf{a}$ and $\mathbf{b}$ as defined above

$$\hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}} = (\mathbf{a}+\mathbf{b}) \,. \tag{24}$$

Proof: The product $\hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}}$ is the geometric product of three vectors, and thus has vector and trivectors parts, generally. However, since we are considering this product in 2-dimensional space, its trivector part is identically zero. Thus $\hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}}$ must be a vector. Now, let

$$\mathbf{V} \equiv \hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}} \,. \tag{25}$$

On taking the reversion of both sides of this last equation, we get

$$\mathbf{V}^\dagger = \mathbf{V} = \hat{\mathbf{b}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{a}}\,. \tag{26}$$

On multiplying the corresponding sides of these last equations, we get

$$\mathbf{V}^2 = [\hat{\mathbf{b}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{a}}][\hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}}] = (\mathbf{a}+\mathbf{b})^2\,. \tag{27}$$

Hence,

$$\mathbf{V} = \pm(\mathbf{a}+\mathbf{b})\,, \tag{28}$$

where we must choose the plus sign. Therefore,

$$\hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}} = (\mathbf{a}+\mathbf{b})\,. \tag{29}$$

From Penn's complex solution to my vector solution

Now it's time to transform Eq. (1) to Eq. (10). Our first step is to rewrite (1) into the form

$$\gamma(\alpha+\beta) = 2\alpha\beta\,. \tag{30}$$

We can virtually emplace $\boldsymbol{\sigma}_1^2 = 1$ in a couple useful places:

$$\gamma\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1(\alpha+\beta) = 2\alpha\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1\beta\,, \tag{31}$$

which gives us

$$\gamma\boldsymbol{\sigma}_1(\mathbf{a}+\mathbf{b}) = 2\alpha\boldsymbol{\sigma}_1\mathbf{b}\,. \tag{32}$$

Next, we multiply this on the left by $\boldsymbol{\sigma}_1$ and use (20) to get

$$\gamma^\dagger(\mathbf{a}+\mathbf{b}) = 2\alpha^\dagger\mathbf{b}\,. \tag{33}$$

On taking the reversion of both sides, we have that

$$(\mathbf{a}+\mathbf{b})\gamma = 2\mathbf{b}\alpha\,. \tag{34}$$

Now we multiply on the left by $(\mathbf{a}+\mathbf{b})$:

$$(\mathbf{a}+\mathbf{b})^2\gamma = 2(\mathbf{a}+\mathbf{b})\mathbf{b}\alpha\,. \tag{35}$$

Solving this for γ and using (23) and (21), we get

$$\gamma = \frac{\alpha(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}}}{R(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})}\,. \tag{36}$$

Multiplying on the left by $\boldsymbol{\sigma}_1$, and using that $\boldsymbol{\sigma}_1\gamma = \mathbf{c}$ and $\boldsymbol{\sigma}_1\alpha = \mathbf{a}$, we get

$$\mathbf{c} = \frac{\mathbf{a}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}}}{R(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})} = \frac{\hat{\mathbf{a}}(\mathbf{a}+\mathbf{b})\hat{\mathbf{b}}}{1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}} = \frac{\mathbf{a}+\mathbf{b}}{1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}}\,, \tag{37}$$

where we also used (24).

Compare the solutions for a specific problem

Let's try the case when $\alpha = i$ and $\beta = 1$. Then

$$\gamma = \frac{2\alpha\beta}{\alpha + \beta} = \frac{2i}{i+1} = 1 + i\,. \tag{38}$$

However, to convert to the equivalent vectors, we have that $\alpha = i \rightarrow \mathbf{a} = \boldsymbol{\sigma}_2$ and $\beta = 1 \rightarrow \mathbf{b} = \boldsymbol{\sigma}_1$. Hence,

$$\mathbf{c} = \frac{\mathbf{a} + \mathbf{b}}{1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}} = \frac{\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_1}{1 + 0} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\,. \tag{39}$$

This is the same result we would get if we multiplied (38) through by $\boldsymbol{\sigma}_1$ and interpreted i as $\mathbf{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$.