Notes on *The Design of Linear Algebra and Geometry*

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Abstract

This paper contains my personal notes on the paper *The Design of Linear Algebra and Geometry*.[1] My comments are meant 1) to clarify certain parts of the exposition (especially for readers, like myself, who are not experts in projective or conformal geometry), 2) to fill-in some of the steps in the mathematical derivations, and 3) to report on a few mistakes that may remain in the preprint version of the paper. As a word of warning, this paper will make no attempt to teach the full fundamentals of geometric/Clifford algebra, though it will spend some time enhancing the discussion on it presented in the paper.

1 Introduction

This paper is the third of a series of papers on projective geometry (and now linear algebra) papers written by D. Hestenes and his coauthors. The first paper was *Projective Geometry with Clifford Algebra*,[2] The second paper was D. Hestenes, Universal Geometric Algebra, Quarterly Jur. of Pure and Applied Mathematics, Simon Stevin 62, 253–274, (September – December, 1988). These papers were published in the late 1980s and early 1990s. The reader should consider these two papers, especially the geometric algebra part, as prerequisites for this paper.

I will be referencing both the published version of the present article and its preprint version. It is not my purpose to present a full introduction to geometric algebra in these notes. However, I will try to flesh-out some of the steps to the equations that have been left to the reader to provide. And, in particular, I will skip over those aspects of this paper that are well trodden in the first two papers of this series. After a brief introduction, that will take us to page 68 in the published version, which is page 4 of the preprint.

[1] D. Hestenes, *The Design of Linear Algebra and Geometry*, *Acta Appl. Math.* Vol. 23 , 65–93 (1991.

[2] D. Hestenes, R. Ziegler, *Projective Geometry with Clifford Algebra*, *Acta Appl. Math.* Vol. 23, 25–63 (1991).

2 Outermorphisms

Once we have defined a linear transformation f on a vector space, we have a natural way to extend that transformation $\underline{f}$ onto all elements of the geometric algebra over that vector space. The basic form it takes is

$$\underline{f}(A+B) = \underline{f}(A) + \underline{f}(B)\,, \tag{1}$$

which is Eq. (2.1), together with

$$\underline{f}(a_1 \wedge a_2 \wedge \cdots \wedge a_r) = \underline{f}(a_1) \wedge \underline{f}(a_1) \wedge \cdots \wedge \underline{f}(a_r)\,, \tag{2}$$

Now, to every linear transformation $\underline{f}$ on a vector space, there exists a corresponding transformation $\bar{f}$ (called the *adjoint* of f) that goes in the opposite direction, that satisfies the relation

$$\langle\, B\underline{f}A\,\rangle = \langle\,(\bar{f}B)A\,\rangle\,, \tag{3}$$

An equation we'll need is (2.15)

$$\underline{f}^{-1}A = \frac{\bar{f}(AI)I^{-1}}{\det f} = \frac{I^{-1}\bar{f}(AI)}{\det f}\,. \tag{4}$$

Now, Eq. (2.17) is new to this series, so I'll start with it. Let's begin with Eq. (2.13):

$$A\cdot(\underline{f}B) = \underline{f}[\,(\bar{f}A)\cdot B\,] \quad \text{or} \quad (\underline{f}B)\cdot A = \underline{f}[\,B\cdot(\bar{f}A)\,]\,. \tag{5}$$

First, we swap the under bars with the over bars, to get

$$A\cdot(\bar{f}B) = \bar{f}[\,(\underline{f}A)\cdot B\,] \quad \text{or} \quad (\bar{f}B)\cdot A = \bar{f}[\,B\cdot(\underline{f}A)\,]\,. \tag{6}$$

Second, we replace B with I^{-1}, to get

$$A\cdot(\bar{f}I^{-1}) = \bar{f}[\,(\underline{f}A)\cdot I^{-1}\,] \quad \text{or} \quad (\bar{f}I^{-1})\cdot A = \bar{f}[\,I^{-1}\cdot(\underline{f}A)\,]\,. \tag{7}$$

We want the equation on the LHS of (7), which gives us

$$A\cdot(\det f I^{-1}) = \bar{f}[\,(\underline{f}A)\cdot I^{-1}\,]\,. \tag{8}$$

which becomes

$$(\det f)AI^{-1} = \bar{f}[\,(\underline{f}A)I^{-1}\,]\,. \tag{9}$$

Next, we replace A on the RHS by $\widetilde{A}I$

$$(\det f)AI^{-1} = \bar{f}[\,(\underline{f}\widetilde{A}I)I^{-1}\,]\,. \tag{10}$$

Then,

$$(\det f)A = \bar{f}[\,(\underline{f}\widetilde{A}I)I^{-1}\,]I\,. \tag{11}$$

The equation I'm aming for is (2.19)

$$(\det f)A = \underline{f}[\,(\bar{f}\,\widetilde{A})I\,]\,. \tag{12}$$

In order to get there from (11), we need to show that

$$\bar{f}\,[\,(\underline{f}\widetilde{A}I)I^{-1}\,] = \underline{f}[\,(\bar{f}\,\widetilde{A})I\,]I^{-1}\,. \tag{13}$$

The next significant equation doesn't have an equation number in the original papers, and it is

$$\begin{aligned}
(\underline{f}^{-1}A)\wedge(\underline{f}^{-1}B) &= \frac{[\,\bar{f}\,(AI)I\,]\wedge[\,(\bar{f}\,\widetilde{B})I\,]}{(\det f)^2} = \frac{[\,(\bar{f}\,\widetilde{A})I\,]\cdot(\bar{f}\,\widetilde{B})}{(\det f)^2}I \\
&= \frac{\bar{f}\,\{\,\underline{f}[\,(\bar{f}\,\widetilde{A})I\,]\cdot\widetilde{B}\}}{(\det f)^2}I = \frac{\bar{f}\,[\,A\cdot\widetilde{B}\,]I}{\det f} \\
&= \frac{\bar{f}\,[\,(A\wedge B)I^{-1}\,]}{\det f}\,.
\end{aligned} \tag{14}$$

To follow the proof of this theorem, we need a lemma:

Lemma 1: Let

$$A = \langle\, A\,\rangle_s \quad \text{and} \quad B = \langle\, B\,\rangle_r\,. \tag{15}$$

Then,

$$A\wedge(BI) = (A\cdot B)I\,. \tag{16}$$

$$\begin{aligned}
A\wedge(BI) &= A\wedge\langle\, BI\,\rangle_{n-r} \\
&= \langle\, ABI\,\rangle_{n-r+s} \\
&= \langle\, ABI\,\rangle_{n-(r-s)} \\
&= (A\cdot B)I\,.
\end{aligned} \tag{17}$$

Now, applying this to the numerator of the first term of the first line on the RHS of (14), we get

$$[\,\bar{f}\,(AI)I\,]\wedge[\,(\bar{f}\,\widetilde{B})I\,] = \{[\,(\bar{f}\,\widetilde{A})I\,]\cdot(\bar{f}\,\widetilde{B})\}I\,. \tag{18}$$

To follow the proof of Eq. (2.21), we need some definitions. First up is for the *meet* $\vee$ of two objects:

$$A\vee B \equiv \widetilde{A}\cdot B\,, \tag{19}$$

where

$$\widetilde{A} \equiv AI^{-1}\,, \tag{20}$$

3 Invariant Blades

We define an *eigenblade* of $\underline{f}$ to arise in the case that

$$\underline{f}A = \lambda A\,, \tag{21}$$

where λ is a real number. In the projective sense, the blade A in (21) is 'fixed' by the transformation, as we can ignore the specific value of λ. Now, to those who are used to dealing with the eigenvectors and eigenvalues of vectors due to the action of a linear transformation, ignoring the magnitudes of the eigenvalues may seem terribly strange, but that is the nature of, and the main virtue of, projective geometry. An eigenblade is said to be *symmetric* if the action on it by $\underline{f}$ is the same as the action of $\bar{f}$ on it.

Now, we already know that for a unit pseudoscalar I:

$$\underline{f}I = \bar{f}\,I = (\det f)I \equiv \mu I\,. \tag{22}$$

We may well ask if the fact that a particular blade A is an eigenblade of $\underline{f}$ whether or not $\widetilde{A}$ is an eigenblade of $\bar{f}$, where $\widetilde{A} = AI^{-1}$.

We need Eq. (2.14) for this proof. Start with Eq. (2.13) and interchange $\bar{f}$ with $\underline{f}$:

$$A\cdot(\bar{f}\,B) = \bar{f}\,[\,(\underline{f}A)\cdot B\,] \quad \text{or} \quad (\bar{f}\,B)\cdot A = \bar{f}\,[\,B\cdot(\underline{f}A)\,]\,. \tag{23}$$

Now, replace B by the unit pseudoscalar I:

$$A(\bar{f}\,I) = \bar{f}\,[\,(\underline{f}A)I\,] \quad \text{or} \quad (\bar{f}\,I)A = \bar{f}\,[\,I(\underline{f}A)\,]\,, \tag{24}$$

which is Eq. (2.14). So, let's take the LHS of the last equations, replacing $\underline{f}A = \lambda A$ and then replace I by I^{-1}:

$$A(\bar{f}\,I^{-1}) = \bar{f}\,[\,(\lambda A)I^{-1}\,]\,, \tag{25}$$

which becomes

$$(\det f)\widetilde{A} = \lambda\bar{f}\,(\widetilde{A})\,. \tag{26}$$

On rearranging and using (22), we get

$$\bar{f}\,(\widetilde{A}) = \frac{\mu}{\lambda}\widetilde{A}\,, \tag{27}$$

which is Eq. (3.4).

Given that

$$I = A^{-1}\widetilde{A} = A^{-1}\wedge\widetilde{A}\,, \tag{28}$$

and that $\mathrm{step}(I) = n$, $\mathrm{step}(A) = r$, and $\mathrm{step}(\widetilde{A}) = n - r$, then

$$\bar{f}\,(I) = (\bar{f}\,A^{-1})\wedge\bar{f}\,(\widetilde{A}) = \frac{\mu}{\lambda}(\bar{f}\,A^{-1})\wedge\widetilde{A}\,, \tag{29}$$

where we used (27). But, since $\bar{f}(I) = \mu I$, then this last equation can be written as

$$I = \frac{1}{\lambda}(\bar{f}\,A^{-1}) \wedge \widetilde{A}\,. \tag{30}$$

On comparing this to (28), we can set $I = X \wedge \widetilde{A}$. We can try the ansatz

$$X = \lambda A + B\,, \tag{31}$$

where

$$B \wedge \widetilde{A} = 0\,. \tag{32}$$

On substitution, we have that

$$I = A \wedge \widetilde{A}\,, \tag{33}$$

hence,

$$\bar{f}\,A^{-1} = A + \frac{1}{\lambda}B\,. \tag{34}$$

For the time being, I will skip over the rest of this section to move ahead into more general topics.

4 Projective Splits

At this point, I will content myself to just reproduce the notes I made for the similar part of the article *Universal Geometric Algebra* [3], at least for the first part of this section.

Earlier, we saw the points of $\mathcal{P}_2$ as represented by vectors in $\mathcal{V}_3$. Now we'll generalize this to $\mathcal{P}_n$ $(\mathcal{V}_n)$ as embedded in the vector space $\mathcal{V}_{n+1}$.

We'll construct an algebraic relationship between $\mathcal{V}_{n+1}$ and $\mathcal{V}_n$. We now define the set of 'vectors'

$$\mathcal{V}_n = \{x \wedge \mathbf{e}_0 \,|\, x \in V_{n+1}\}\,, \tag{35}$$

which is Eq. (34) in the preprint paper.

If you take the geometric algebra of this, $\mathcal{G}_n$, you get first the bivectors of $\mathcal{V}_{n+1}$, of course. If you take inner products of bivectors, you get scalars. If you combine the bivectors to make higher-graded objects, you get 4-vectors, 6-vectors, etc. In other words, all that taken together gives us the set of even elements of $\mathcal{G}_{n+1}$. But does this set have an algebraic structure? It does. Since the product of any two even elements of $\mathcal{G}_{n+1}$ is another even element of $\mathcal{G}_{n+1}$, then the set constitutes the even subalgebra of $\mathcal{G}_{n+1}$.

But what about the set of 'vectors' we defined in (35)? Is this really a 'vector' space? It certainly is under the usual definition of a vector space. It has a zero vector, namely $\mathbf{e}_0 \wedge \mathbf{e}_0 = 0$. It's closed under addition of 'vectors' and under scalar multiplication, etc.

But wait! If $\mathcal{V}_n$ as defined in (35) is a legitimate vector space, then it should have its own geometric algebra $\mathcal{G}_n$, right? Right.

Anyway, back to analyzing a typical element of $\mathcal{V}_n$, that being $x \wedge \mathbf{e}_0$. Let's think about this in $\mathcal{P}_2$, which is the projective plane in $\mathcal{V}_3$. We said that if a and b are any two distinct points in $\mathcal{P}_2$, that we can represent the join of these points as $a \wedge b$. This join is a line in $\mathcal{P}_2$ containing points a and b. Furthermore, $a \wedge b$ is a 2-blade in $\mathcal{G}_3$.

One way to think of $\mathcal{P}_2$ that contains the point $\mathbf{e}_0$ is that it is the set of all points in $\mathcal{P}_2$ whose joins with $\mathbf{e}_0$ are orthogonal to $\mathbf{e}_0$. The text claims that $x \wedge \mathbf{e}_0$ is a linear map from $\mathcal{V}_{n+1}$ to V_n. Let's investigate this a bit more formally. Let L be a map from $\mathcal{V}_{n+1}$ to $\mathcal{V}_n$,

$$L : V_{n+1} \to \mathcal{V}_n \quad \text{given by} \quad L(x) = x \wedge \mathbf{e}_0 \, . \tag{36}$$

Show that this mapping is linear. Let α be a scalar, then,

$$L(\alpha x) = (\alpha x) \wedge \mathbf{e}_0 = \alpha(x \wedge \mathbf{e}_0) = \alpha L(x) \, . \tag{37}$$

So, it treats scalars properly. What about vector addition?

$$L(x+y) = (x+y) \wedge \mathbf{e}_0 = x \wedge \mathbf{e}_0 + y \wedge \mathbf{e}_0 = L(x) + L(y) \, . \tag{38}$$

And it treats vector addition properly, hence, it's a linear map.

We can give the elements of $\mathcal{V}_n$ a cosmetic upgrade by letting $x_0 = x \cdot \mathbf{e}_0 \in \mathbb{R}$ and $\mathbf{x} \equiv x \wedge \mathbf{e}_0 / x \cdot \mathbf{e}_0$ for each $x \in \mathcal{V}_{n+1}$, then

$$x\mathbf{e}_0 = x \cdot \mathbf{e}_0 + x \wedge \mathbf{e}_0 = x_0(1 + \mathbf{x}) \, , \tag{39}$$

which is Eq. (36) in the preprint paper.

Lemma:

$$\mathbf{e}_0 x = x_0(1 - \mathbf{x}) \, . \tag{40}$$

Proof:

$$\mathbf{e}_0 x = (x\mathbf{e}_0)^\dagger = [\, x_0(1 + x \wedge \mathbf{e}_0)\,]^\dagger = x_0(1 - x \wedge \mathbf{e}_0) = x_0(1 - \mathbf{x}) \, , \tag{41}$$

where $x_0 = x \cdot e_0$ and $\mathbf{x} = x \wedge e_0 / x \cdot e_0$.

We'll now prove Eq. (37), which is

$$a \wedge b = a_0 b_0 (\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}) = a_0 b_0 (\mathbf{u} + \mathbf{a} \wedge \mathbf{u}) \, , \tag{42}$$

where $\mathbf{e}_0^2 = 1$, $a\mathbf{e}_0 = a_0(1 + \mathbf{a})$, and $b\mathbf{e}_0 = b_0(1 + \mathbf{b})$.

So, for the proof:

$$\begin{aligned}
a \wedge b &= \tfrac{1}{2}(ab - ba) \\
&= \tfrac{1}{2}(a\mathbf{e}_0\mathbf{e}_0 b - b\mathbf{e}_0\mathbf{e}_0 a) \\
&= \tfrac{1}{2}[\,(a\mathbf{e}_0)(\mathbf{e}_0 b) - (b\mathbf{e}_0)(\mathbf{e}_0 a)\,] \\
&= \tfrac{1}{2}[\,a_0 b_0(1+\mathbf{a})(1-\mathbf{b}) - a_0 b_0(1+\mathbf{b})(1-\mathbf{a})\,] \\
&= a_0 b_0[\,\mathbf{a} - \mathbf{b} + \tfrac{1}{2}(\mathbf{ba} - \mathbf{ba})\,] \\
&= a_0 b_0[\,\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}\,]\,.
\end{aligned} \tag{43}$$

If we let $\mathbf{u} \equiv \mathbf{a} - \mathbf{b}$ and $\mathbf{M} \equiv \mathbf{a} \wedge \mathbf{u} = \mathbf{b} \wedge \mathbf{a}$, this last result can be written as

$$a \wedge b = a_0 b_0(\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}) = a_0 b_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,. \tag{44}$$

Lemma:

$$\mathbf{a} \wedge \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{b} \wedge \mathbf{a} = -\mathbf{c} \cdot \mathbf{a} \wedge \mathbf{b}\,, \tag{45}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors.

Proof:

$$\langle\,\mathbf{abc}\,\rangle_1 = \langle\,\mathbf{abc}\,\rangle_1^\dagger = \langle\,\mathbf{cba}\,\rangle_1\,.$$

Expanding both sides,

$$\langle\,\mathbf{a} \cdot \mathbf{bc} + \mathbf{a} \wedge \mathbf{bc}\,\rangle_1 = \langle\,\mathbf{cb} \cdot \mathbf{a} + \mathbf{cb} \wedge \mathbf{a}\,\rangle_1\,.$$

On dropping a term, gives

$$\langle\,\mathbf{a} \wedge \mathbf{bc}\,\rangle_1 = \langle\,\mathbf{cb} \wedge \mathbf{a}\,\rangle_1\,.$$

Hence,

$$\mathbf{a} \wedge \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{b} \wedge \mathbf{a} = -\mathbf{c} \cdot \mathbf{a} \wedge \mathbf{b}\,, \tag{46}$$

Now, on to Eq. (38), which is a bit more involved.

$$x \wedge a \wedge b = x_0 a_0 b_0[\,(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x}\,\mathbf{a} \wedge \mathbf{u})\mathbf{e}_0 = 0\,. \tag{47}$$

The reason this quantity is zero is by design, since we are looking for all x that lie in the plane described by the 2-blade $a \wedge b$. So, we begin:

Let $B \equiv a \wedge b$, then $x \wedge a \wedge b$ can be expressed as $x \wedge B$. Hence,

$$x \wedge B = \tfrac{1}{2}(xB + Bx)\,. \tag{48}$$

Now, it's time to be a little bit tricky in how we introduce the projective split. We'll start by multiplication on the right by $\mathbf{e}_0$:

$$2x \wedge B\mathbf{e}_0 = xB\mathbf{e}_0 + Bx\mathbf{e}_0\,. \tag{49}$$

We already have an expression for $B = a \wedge b$ in (44), therefore,

$$\begin{aligned} 2x \wedge B\mathbf{e}_0 = &\; xB\mathbf{e}_0 + Bx\mathbf{e}_0 \\ &= x[\, a_0 b_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,]\mathbf{e}_0 + [\, a_0 b_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,]x\mathbf{e}_0\,. \end{aligned} \tag{50}$$

Let $\Omega = 2x \wedge B\mathbf{e}_0 / x_0 a_0 b_0$, then (with $\mathbf{e}_0^2 = 1$)

$$\begin{aligned} \Omega x_0 &= x[\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0 + [\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]x\mathbf{e}_0 \\ &= x\mathbf{e}_0\mathbf{e}_0[\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0 + [\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]x\mathbf{e}_0 \\ &= x_0(1 + \mathbf{x})\{\mathbf{e}_0[\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0\} + [\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]x_0(1 + \mathbf{x})\,. \end{aligned} \tag{51}$$

Therefore, some simplification yields

$$\Omega = (1 + \mathbf{x})\{\mathbf{e}_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\mathbf{e}_0\} + (\mathbf{u} + \mathbf{a} \wedge \mathbf{u})(1 + \mathbf{x})\,. \tag{52}$$

So, now everything hinges on how we can get rid of the $\mathbf{e}_0$'s in the first term on the RHS. With the understanding that the vectors $\mathbf{a}$ and $\mathbf{u}$ are orthogonal to $\mathbf{e}_0$, and that $\mathbf{e}^2 = 1$, we get

$$\begin{aligned} \mathbf{e}_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\mathbf{e}_0 &= \mathbf{e}_0\mathbf{u}\mathbf{e}_0 + \mathbf{e}_0\mathbf{a} \wedge \mathbf{u}\mathbf{e}_0 \\ &= (2\mathbf{e}_0 \cdot \mathbf{u} - \mathbf{u}\mathbf{e}_0)\mathbf{e}_0 + \langle\, \mathbf{e}_0\mathbf{a} \wedge \mathbf{u}\mathbf{e}_0 \,\rangle_2 \\ &= -\mathbf{u} + \mathbf{e}_0 \cdot (\mathbf{a} \wedge \mathbf{u} \wedge \mathbf{e}_0) \\ &= -\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,. \end{aligned} \tag{53}$$

On substituting this result into (52), we get

$$\begin{aligned} \Omega &= (1 + \mathbf{x})(-\mathbf{u} + \mathbf{a} \wedge \mathbf{u}) + (\mathbf{u} + \mathbf{a} \wedge \mathbf{u})(1 + \mathbf{x}) \\ &= -\mathbf{u} + \mathbf{a} \wedge \mathbf{u} - \mathbf{x}\mathbf{u} + \mathbf{x} \cdot \mathbf{a} \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} + \mathbf{u} \\ &\qquad + \mathbf{a} \wedge \mathbf{u} + \mathbf{a} \wedge \mathbf{x} + \mathbf{a} \wedge \mathbf{u} \cdot \mathbf{x} \\ &= 2\mathbf{a} \wedge \mathbf{u} + (\mathbf{u}\mathbf{x} - \mathbf{x}\mathbf{u}) + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} \\ &= 2\mathbf{a} \wedge \mathbf{u} + 2\mathbf{u} \wedge \mathbf{x} + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} \\ &= 2(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,, \end{aligned} \tag{54}$$

where, on going between steps 2 and 3, we did a lot of cancellation, using, in particular, (45). Hence, we have (50) becoming

$$2x \wedge B\mathbf{e}_0 / x_0 a_0 b_0 = 2(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,. \tag{55}$$

From this we get

$$x \wedge B = x_0 a_0 b_0[\,(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0\,. \tag{56}$$

Using that $B = a \wedge b$, we have that

$$x \wedge a \wedge b = x_0 a_0 b_0[\,(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0\,. \tag{57}$$

For $x \wedge a \wedge b$ to vanish, we need

$$(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} = 0\,. \tag{58}$$

Now we have arrived at a fun part of the paper: the part that proves the invariance of the **cross ratio**. So, we start with three distinct points a, b, c on a given line in $\mathcal{P}_2$. (We deduce that the wedge product of any two of them is a nonzero scalar multiple of the wedge product of any other two of them.)

So, if we can show that

$$b_0 a \wedge c(\mathbf{b} - \mathbf{c}) = a_0(b \wedge c)(\mathbf{a} - \mathbf{c})\,, \tag{59}$$

then we can write

$$\frac{a \wedge c}{b \wedge c} = \frac{a_0(\mathbf{a} - \mathbf{c})}{b_0(\mathbf{b} - \mathbf{c})} = \frac{a_0}{b_0}\frac{\mathbf{a} - \mathbf{c}}{\mathbf{b} - \mathbf{c}}\,, \tag{60}$$

which is Eq. (39) in the preprint paper.

In preparation, we need a couple results first. For one, $\mathbf{b} - \mathbf{c}$ is related to $\mathbf{a} - \mathbf{c}$ by a factor of a nonzero scalar multiple,[3] say α, or

$$\mathbf{b} - \mathbf{c} = \alpha(\mathbf{a} - \mathbf{c})\,. \tag{61}$$

Now, on wedging this last result by $\mathbf{c}$ on the left, we get the next result

$$\mathbf{c} \wedge \mathbf{b} = \alpha \mathbf{c} \wedge \mathbf{a}\,. \tag{62}$$

We also need the following lemma. Starting with

$$a \wedge c = a_0 c_0(\mathbf{a} - \mathbf{c} + \mathbf{c} \wedge \mathbf{a})\,, \tag{63}$$

we get that

$$\begin{aligned}
\alpha(a \wedge c) &= a_0 c_0(\alpha(\mathbf{a} - \mathbf{c}) + \alpha \mathbf{c} \wedge \mathbf{a}) \\
&= a_0 c_0((\mathbf{b} - \mathbf{c}) + \mathbf{c} \wedge \mathbf{b}) \\
&= \frac{a_0}{b_0} b_0 c_0((\mathbf{b} - \mathbf{c}) + \mathbf{c} \wedge \mathbf{b}) \\
&= \frac{a_0}{b_0} b \wedge c\,.
\end{aligned} \tag{64}$$

So, let's start on the LHS of (59) and proceed to the RHS.

$$\begin{aligned}
b_0 a \wedge c\,(\mathbf{b} - \mathbf{c}) &= a \wedge c\, b_0 \alpha(\mathbf{a} - \mathbf{c}) \\
&= \alpha(a \wedge c)\, b_0(\mathbf{a} - \mathbf{c}) \\
&= \frac{a_0}{b_0} b \wedge c\, b_0(\mathbf{a} - \mathbf{c}) \quad \text{(using (64))} \\
&= b \wedge c\, a_0(\mathbf{a} - \mathbf{c}) \\
&= a_0(b \wedge c)(\mathbf{a} - \mathbf{c})\,.
\end{aligned} \tag{65}$$

[3]This is because points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ lie on the same line.

However, Eq. (60) has the scalars a_0 and b_0, thus this relation, being based on only three points, is not the invariant relationship we seek. To find it, let's substitute d for c (where d is yet another distinct point on the same line) in (60) to get

$$\frac{a \wedge d}{b \wedge d} = \frac{a_0(\mathbf{a} - \mathbf{d})}{b_0(\mathbf{b} - \mathbf{d})} = \frac{a_0}{b_0}\frac{\mathbf{a} - \mathbf{d}}{\mathbf{b} - \mathbf{d}}\,, \tag{66}$$

If we now divide (60) by (66) we get

$$\frac{a \wedge c}{b \wedge c}\frac{b \wedge d}{a \wedge d} = \frac{\mathbf{a} - \mathbf{c}}{\mathbf{b} - \mathbf{c}}\frac{\mathbf{b} - \mathbf{d}}{\mathbf{a} - \mathbf{d}}\,, \tag{67}$$

which is Eq. (40) of the preprint paper and is also the invariant cross ratio, based on four distinct points.

Next. we examine translations. We pick a direction e_0 in $\mathcal{V}_{n+1}$ to determine a family of hyperplanes by $x \cdot e_0 = \lambda$. Given that our translation operator $\underline{T}_a$ is given by

$$\underline{T}_a x = x + ax \cdot e_0 = x + ax \cdot e_0\,, \tag{68}$$

determine the operator $\bar{T}_a x$, subject to the constraints

$$\bar{T}_a e_0 = e_0 \quad \text{and} \quad a \cdot e_0 = 0\,. \tag{69}$$

The general equation that relates linear transformations and their adjoints is

$$\langle\, (\bar{T}_a y)x \,\rangle = \langle\, y \underline{T}_a x \,\rangle\,. \tag{70}$$

Let's try an ansatz for $\bar{T}_a$ consistent with (69), namely,

$$\bar{T}_a y = \alpha y + \beta e_0\,. \tag{71}$$

Then, we get

$$\langle\, (\alpha y + \beta e_0)x \,\rangle = \langle\, y(x + ax \cdot e_0) \,\rangle\,, \tag{72}$$

or, rather,

$$\alpha y \cdot x + \beta e_0 \cdot x = y \cdot x + y \cdot ax \cdot e_0\,. \tag{73}$$

From this we get

$$\alpha = 1\,, \quad \beta = a \cdot y\,. \tag{74}$$

Therefore, we can rewrite (71) in terms of x as

$$\bar{T}_a x = x + a \cdot x\, e_0\,, \tag{75}$$

which is Eq. (4.20).

It's straightforward to produce the equation between (4.20) and (4.21).

$$\begin{aligned} \underline{T}_a(x \wedge y) &= \underline{T}_a(x) \wedge \underline{T}_a(y) \\ &= [\, x + ax \cdot e_0 \,] \wedge [\, y + ay \cdot e_0 \,] \\ &= x \wedge y + a \wedge (x \cdot e_0 y - y \cdot e_0 x)\,. \end{aligned} \tag{76}$$

The last equation I want to establish before going on to the next section is Eq. (4.32).

S_0^2 is a rotor that rotates, without rescaling, the vector e_0 to v, and both of these vectors are unit vectors.

$$v = S_0^2 e_0 \,. \tag{77}$$

Then, S_0 rotates e_0 through half that angle. Let's refer to the resultant vector as u.

$$u = S_0 e_0 \,, \tag{78}$$

It's easy to see that the vector u shares the same direction as $e_0 + v$, since both these vectors have the same length and thus their sum results in a vector half-way between them. If we divide this vector by its magnitude, we get a unit vector. Therefore,

$$u = S_0 e_0 = \frac{e_0 + v}{|\, e_0 + v \,|} = \frac{v + e_0}{|\, e_0 + v \,|} \,. \tag{79}$$

On solving this for S_0, we get

$$S_0 = \frac{(v + e_0)e_0}{|\, e_0 + v \,|} = \frac{ve_0 + 1}{|\, e_0 + v \,|} = \frac{ve_0 + v^2}{|\, e_0 + v \,|} = \frac{v(v + e_0)}{|\, e_0 + v \,|} \,, \tag{80}$$

which is Eq. (4.32). Finally, solving for S_0 from (77), we get

$$S_0 = (ve_0)^{1/2} \,. \tag{81}$$

5 Conformal and Metric Geometry

There's not a lot of computation to do in the first subsection. To me, the most interesting equation to deal with in this subsection is (5.20), which is

$$M = \tfrac{1}{2}[\, A(1 + e_0) + B(e_1 + e_2) + C(e_1 - e_2) + D(1 - e_0) \,] \,, \tag{82}$$

where the coefficients are scalars, and

$$[\, e_1 \,] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad [\, e_2 \,] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \qquad [\, e_0 \,] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{83}$$

On performing the sums or differences in (82), we get the four basis elements

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \tag{84}$$

of a four-dimensional vector space over the real numbers.

$\vdash$ Our next big goal is to establish Eq. (5.44).

We start with a vector basis for $\mathcal{V}_2 = \mathcal{V}(1,1)$, namely, $\{e_1, e_2\}$, where

$$e_1^2 = 1\,, \qquad e_2^2 = -1\,, \qquad e_1 \cdot e_2 = 0\,, \tag{85}$$

and these properties have been chosen as much for convenience as for anything else. In this basis we can choose

$$e_0 \equiv e_1 \wedge e_2\,, \tag{86}$$

where e_0 is a 2-blade of $\mathcal{G}^2_{n+2}$. From which we derive that

$$e_0^2 = 1\,. \tag{87}$$

Now, the magic occurs when we adopt a null basis set for $\mathcal{V}(1,1)$, $\{e_+, e_-\}$, where

$$e_\pm \equiv \tfrac{1}{2}(e_1 \pm e_2)\,. \tag{88}$$

It's easy to show that $e_\pm^2 = 0$, and this is possible because of the mixed signatures of e_1 and e_2. But what about $e_+ \cdot e_-$ and $e_+ \wedge e_-$? With a little algebra, one can show that

$$e_+ \cdot e_- = \tfrac{1}{2}\,, \qquad e_+ \wedge e_- = \tfrac{1}{2} e_0\,. \tag{89}$$

On using this latter equation, we have that

$$x \cdot e_0 = 2x \cdot e_+ \wedge e_- = 2(x \cdot e_+ e_- - e_+ x \cdot e_-)\,. \tag{90}$$

Now, we will square this last equation, remembering that $x \cdot e_+$ and $x \cdot e_-$ are scalars, to get

$$(x \cdot e_0)^2 = -4x \cdot e_+ x \cdot e_-\,. \tag{91}$$

Okay, the fundamental expansion of xe_0 is given by

$$xe_0 = x \cdot e_0 + x \wedge e_0\,. \tag{92}$$

On taking the reverse of (92), we get

$$-e_0 x = x \cdot e_0 - x \wedge e_0\,. \tag{93}$$

So,

$$e_0 x = -x \cdot e_0 + x \wedge e_0\,. \tag{94}$$

Therefore,

$$\begin{aligned} x^2 &= (xe_0)(e_0 x) = (x \cdot e_0 + x \wedge e_0)(-x \cdot e_0 + x \wedge e_0) \\ &= -(x \cdot e_0)^2 + x \cdot e_0 x \wedge e_0 - x \wedge e_0 x \cdot e_0 + (x \wedge e_0)^2 \\ &= -(x \cdot e_0)^2 + (x \wedge e_0)^2 \\ &= 0\,, \end{aligned} \tag{95}$$

where

$$x \cdot e_0 x \wedge e_0 - x \wedge e_0 x \cdot e_0 = 0\,, \tag{96}$$

because $x \cdot e_0$ is a vector and $x \wedge e_0$ is a trivector of a 3-D subspace that acts as a pseudoscalar, which then commutes with $x \cdot e_0$. We now define

$$x_0 = x \cdot e_0 \qquad \text{and} \qquad \rho\,\mathbf{x} = x \wedge e_0\,, \tag{97}$$

where we will solve for ρ in due time.

Anyway, we have that

$$(x \cdot e_0)^2 = (x \wedge e_0)^2 = \rho^2 \mathbf{x}^2\,. \tag{98}$$

Our goal now will be to rewrite all products in terms of e_+ and e_- instead e_0. Thus,

$$-4x \cdot e_+ x \cdot e_- = \rho^2 \mathbf{x}^2\,. \tag{99}$$

Our choice for ρ will be

$$\rho = 2x \cdot e_+\,. \tag{100}$$

With this choice, then

$$x \cdot e_- = -\mathbf{x}^2 x \cdot e_+\,. \tag{101}$$

Returning to (92), we get the *conformal split* of xe_0

$$\begin{aligned} xe_0 &= x \cdot e_0 + x \wedge e_0 \\ &= 2[\,(x \cdot e_+)e_- - (x \cdot e_-)e_+\,] + x \wedge e_0 \quad \text{(using (90))} \\ &= 2[\,(x \cdot e_+)e_- - (-\mathbf{x}^2 x \cdot e_+)e_+\,] + x \wedge e_0 \quad \text{(using (101))} \\ &= 2x \cdot e_+ e_- + 2\mathbf{x}^2 x \cdot e_+ e_+ + x \wedge e_0 \\ &= 2x \cdot e_+ e_- + 2\mathbf{x}^2 x \cdot e_+ e_+ + \rho\mathbf{x} \quad \text{(using (98))} \\ &= 2x \cdot e_+ e_- + 2\mathbf{x}^2 x \cdot e_+ e_+ + 2x \cdot e_+ \mathbf{x} \quad \text{(using (100))} \\ &= 2x \cdot e_+(e_- + \mathbf{x}^2 e_+ + \mathbf{x})\,, \end{aligned} \tag{102}$$

where $\mathbf{x} = \dfrac{x \wedge e_0}{x \cdot e_+}$.

The values $x \wedge e_0$ and $x \cdot e_+$ are referred to as the 'homogeneous coordinates' of $\mathbf{x}$, which seems rather strange in that there is no representation of points in our projective space in terms of coordinates in $\mathbb{R}^{n+1}$. Anyway, all our proofs here in projective and conformal geometry use coordinate-free methods.

The story of homogeneous coordinates is interesting and useful. Projective geometry in 2-D used to be performed in the plane. Then, someone thought to add coordinates to that plane, just like in analytic geometry. Then, someone else got the very brilliant idea to raise that plane out of the x, y-plane and embed it into x, y, z-space. Any plane in $\mathbb{R}^3$ would do, just so long as it doesn't contain the origin of coordinates. Since all the algebraic operations on the coordinatized projective plane were just the operations of the Gibbs's vector algebra, then it was found that one could do the projective geometry of the plane using the Gibbs's vector algebra, without using the coordinates. That brings us to now. We upgrade the Gibbs's vector algebra with geometric algebra, which is not only more elegant in 3-D, it generalizes well to n-D.

Now it's time to present a matrix version of the conformal split. Let

$$xe_0 = \rho X\,. \tag{103}$$

Therefore,

$$X = e_- + \mathbf{x}^2 e_+ + \mathbf{x}\,. \tag{104}$$

We can convert this equation to 'matrix form' by allowing arbitrary elements of the geometric algebra as their entries. For example, $\mathbf{x} \to \mathbf{x}\mathbb{I}$, where $\mathbb{I}$ is the unit 2×2 matrix, and

$$e_- \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\,, \qquad e_+ \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\,. \tag{105}$$

Thus,

$$[\,X\,] = \begin{bmatrix} \mathbf{x} & \mathbf{x}^2 \\ 1 & \mathbf{x} \end{bmatrix}\,, \tag{106}$$

which is Eq. (5.46).

At this point, I will jump to establish Eq. (5.26), which is

$$\det[\,M\,] = \widetilde{M}M\,, \tag{107}$$

where

$$\widetilde{M} \equiv (M^*)^\dagger\,, \tag{108}$$

and where

$$M^* \equiv e_0 M e_0\,. \tag{109}$$

We'll need some results in advance.

$$e_0 e_1 = e_1 \wedge e_2 \cdot e_1 = e_1 \cdot e_1 \wedge e_1 = -e_2\,, \tag{110a}$$

$$e_0 e_2 = e_1 \wedge e_2 \cdot e_2 = e_2 \cdot e_1 \wedge e_1 = -e_1\,. \tag{110b}$$

Then

$$e_0 e_1 e_0 = -e_2 e_0 = -e_1\,, \tag{110c}$$

$$e_0 e_2 e_0 = -e_1 e_0 = -e_2\,. \tag{110d}$$

First step, take this Eq. (109) and produce (with $e_0^2 = 1$)

$$\begin{aligned} M^* &= \tfrac{1}{2} e_0 [\,A(1+e_0) + B(e_1+e_2) + C(e_1-e_2) + D(1-e_0)] e_0 \\ &= \tfrac{1}{2}[\,A(1+e_0) + B(e_0e_1e_0 + e_0e_2e_0) + C(e_0e_1e_0 - e_0e_2e_0) + D(1-e_0)] \\ &= \tfrac{1}{2}[\,A(1+e_0) - B(e_1+e_2) - C(e_1-e_2) + D(1-e_0)]\,. \end{aligned} \tag{111}$$

Now we take the reverse of this last equation:

$$\begin{aligned} \widetilde{M} &= \tfrac{1}{2}[\,A(1+e_0) - B(e_1+e_2) - C(e_1-e_2) + D(1-e_0)]^\dagger \\ &= \tfrac{1}{2}[\,A(1-e_0) - B(e_1+e_2) - C(e_1-e_2) + D(1+e_0)] \\ &= \tfrac{1}{2}[\,D(1+e_0) - B(e_1+e_2) - C(e_1-e_2) + A(1-e_0)]\,. \end{aligned} \tag{112}$$

So, with the conversion of M to $[M]$ according to (5.22):

$$[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} , \tag{113}$$

and

$$[\widetilde{M}] = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} , \tag{114}$$

then

$$\begin{aligned} [\widetilde{M}][M] &= \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} AD - BC & 0 \\ 0 & AD - BC \end{bmatrix} = \det[M] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \end{aligned} \tag{115}$$

My next and last foray into this current paper, at least for the time being, is to justify Eq. (5.59). We begin with Eq. (5.10):

$$x' = Gx(G^*)^{-1} , \tag{116}$$

which gives us

$$GX\hat{G} = \sigma X' , \tag{117}$$

which is Eq. (5.48). Using (102), we get that

$$G(e_- + \mathbf{x}^2 e_+ + \mathbf{x})\hat{G} = \sigma[\, e_- + \mathbf{x}'^2 e_+ + \mathbf{x}' \,] , \tag{118}$$

where $\mathbf{x}' = g(\mathbf{x})$.

Without loss of generality, we can express G in the form

$$G = Ae_+e_1 + Be_+ + Ce_- + De_-e_1 , \tag{119}$$

where the values of A, B, C, D are in $\mathcal{G}_n$, but this form of G has the virtue that we can still use the matrix form of (113). So, let

$$[\, G \,] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} , \tag{120}$$

Now, with respect to (119), what is $G^\dagger$?

$$\begin{aligned} G^\dagger &= A^\dagger e_1 e_+ + B^\dagger e_+ + C^\dagger e_- + D^\dagger e_1 e_- \\ &= A^\dagger e_- e_1 + B^\dagger e_+ + C^\dagger e_- + D^\dagger e_+ e_1 \\ &= D^\dagger e_+ e_1 + B^\dagger e_+ + C^\dagger e_- + A^\dagger e_- e_1 , \end{aligned} \tag{121}$$

Before we calculate G^*, let's get some basic results down first.

$$\begin{aligned} e_1 e_0 &= -e_0 e_1 \\ e_2 e_0 &= -e_0 e_2 , \end{aligned} \tag{122}$$

$$
\begin{aligned}
e_0e_+e_1e_0 &= e_0[\,\tfrac{1}{2}(e_1+e_2)\,]e_0 \\
&= e_0e_0[\,\tfrac{1}{2}(e_1+e_2)\,] \\
&= e_+e_1\,,
\end{aligned} \tag{123}
$$

$$
\begin{aligned}
e_0e_+e_0 &= -e_0e_0[\,\tfrac{1}{2}(e_1+e_2)\,]e_0 \\
&= -\tfrac{1}{2}(e_1+e_2) \\
&= -e_+\,.
\end{aligned} \tag{124}
$$

So,

$$
\begin{aligned}
G^* &= e_0Ge_0 \\
&= e_0[\,Ae_+e_1 + Be_+ + Ce_- + De_-e_1\,]e_0 \\
&= e_0A^\dagger e_0(e_0e_+e_1e_0) + e_0B^\dagger e_0(e_0e_+e_0) + e_0C^\dagger e_0(e_0e_-e_0) + e_0D^\dagger e_0(e_0e_-e_1e_0) \\
&= A^*(e_0e_+e_1e_0) + B^*(e_0e_+e_0) + C^*(e_0e_-e_0) + D^*(e_0e_-e_1e_0) \\
&= A^*e_+e_1 - B^*e_+ - C^*e_- + D^*e_-e_1\,,
\end{aligned} \tag{125}
$$

Therefore

$$
[G^\dagger] = \begin{bmatrix} D^\dagger & B^\dagger \\ C^\dagger & A^\dagger \end{bmatrix} \qquad \text{and} \qquad [\,G^*\,] = \begin{bmatrix} A^* & -B^* \\ -C^* & D^* \end{bmatrix}, \tag{126}
$$

Now,

$$
[\,GG^\dagger] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D^\dagger & B^\dagger \\ C^\dagger & A^\dagger \end{bmatrix} = \begin{bmatrix} AD^\dagger + BC^\dagger & AB^\dagger + BA^\dagger \\ CD^\dagger + DC^\dagger & CB^\dagger + DA^\dagger \end{bmatrix}, \tag{127}
$$

but we have the requirement expressed in (5.55) that

$$
GG^\dagger = |\,G\,|^2\,, \tag{128}
$$

thus,

$$
[\,GG^\dagger\,] = |\,G\,|^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{129}
$$

We also need to know that

$$
[\,\hat{G}\,] = \begin{bmatrix} \hat{D} & \hat{B} \\ \hat{C} & \hat{A} \end{bmatrix}. \tag{130}
$$

On converting (118) to matrix form, we have that

$$
[\,G\,][\,e_- + \mathbf{x}^2e_+ + \mathbf{x}\,][\,\hat{G}\,] = \sigma[\,e_- + \mathbf{x}'^2e_+ + \mathbf{x}'\,]\,, \tag{131}
$$

or

$$
[\,GX\hat{G}\,] = \sigma \begin{bmatrix} g(\mathbf{x}) & [\,g(\mathbf{x})\,]^2 \\ 1 & g(\mathbf{x}) \end{bmatrix}. \tag{132}
$$

And finally,

$$
\begin{aligned}
[\,GX\hat{G}\,] &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x} & \mathbf{x}^2 \\ 1 & \mathbf{x} \end{bmatrix} \begin{bmatrix} \hat{D} & \hat{B} \\ \hat{C} & \hat{A} \end{bmatrix} \\
&= \begin{bmatrix} (A\mathbf{x}+B) & (A\mathbf{x}^2+B\mathbf{x}) \\ (C\mathbf{x}+D) & (C\mathbf{x}^2+D\mathbf{x}) \end{bmatrix} \begin{bmatrix} \hat{D} & \hat{B} \\ \hat{C} & \hat{A} \end{bmatrix} \\
&= \begin{bmatrix} (A\mathbf{x}+B)\hat{D}+(A\mathbf{x}^2+B\mathbf{x})\hat{C} & (A\mathbf{x}+B)\hat{B}+(A\mathbf{x}^2+B\mathbf{x})\hat{A} \\ (C\mathbf{x}+D)\hat{D}+(C\mathbf{x}^2+D\mathbf{x})\hat{C} & (C\mathbf{x}+D)\hat{B}+(C\mathbf{x}^2+D\mathbf{x})\hat{A} \end{bmatrix} \\
&= \begin{bmatrix} A\mathbf{x}\hat{D}+B\hat{D}+A\mathbf{x}^2\hat{C}+B\mathbf{x}\hat{C} & A\mathbf{x}\hat{B}+B\hat{B}+A\mathbf{x}^2\hat{A}+B\mathbf{x}\hat{A} \\ C\mathbf{x}\hat{D}+D\hat{D}+C\mathbf{x}^2\hat{C}+D\mathbf{x}\hat{C} & C\mathbf{x}\hat{B}+D\hat{B}+C\mathbf{x}^2\hat{A}+D\mathbf{x}\hat{A} \end{bmatrix} \\
&= \begin{bmatrix} (A\mathbf{x}+B)(\hat{D}+\mathbf{x}\hat{C}) & (A\mathbf{x}+B)(\hat{B}+\mathbf{x}\hat{A}) \\ (C\mathbf{x}+D)(\hat{D}+\mathbf{x}\hat{C}) & (C\mathbf{x}+D)(\hat{B}+\mathbf{x}\hat{A}) \end{bmatrix}\,. \qquad (133)
\end{aligned}
$$

6 Conclusion

Hopefully, soon I can return to this paper and flesh it the rest of it.

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