Notes on *Projective Geometry with Clifford Algebra*

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Abstract

This paper contains my personal notes on the paper *Projective Geometry with Clifford Algebra*.[1] My comments are meant 1) to clarify certain parts of the exposition (especially for readers, like myself, who are not experts in projective geometry), 2) to fill-in some of the steps in the mathematical derivations, and 3) to report on a few mistakes that remain in the preprint version of the paper. As a word of warning, this paper will make no attempt to teach the fundamentals of geometric/Clifford algebra, though it will spend some time enhancing the discussion on it presented in the paper.

1 Introduction to the series

This paper is meant to be the first of a series of three papers on projective geometry papers written by D. Hestenes and his coauthors. These papers were published in the late 1980s and early 1990s.

I have to confess that I have had an extraordinarily hard time understanding the simplest things about projective geometry. I think that I have finally penetrated at least the outercasing of the subject by working through the book by H. L. Dorwart *The Geometry of Incidence* [2][2] But even his approach was too difficult to understand until I took the implicit advice of David Hestenes to rewrite it all in the Gibbs' vector algebra:

> Note, for example, that the relation of $\widetilde{A}$ to b and c in (4.2) is exactly that of the conventional vector cross product. Consequently, the mathematical language used here articulates smoothly with standard vector algebra of Gibbs so widely used in physics [10]. [Proj. Geom. with Cliff. Alg. p. 39]

I wrote a number of papers that followed Dorwart's presentation, though recast in my version of coordinate-free Gibbs formulation (well, mostly). I have

[1] David Hestenes and Renatus Ziegler, *Acta Applicadae Mathematicae*, Vol. **23**, (1991) 25–63.

[2] H. L. Dorwart, *The Geometry of Incidence*, Prentice-Hall (1966).

re-presented my formulation of how to represent points in the projective plane as vectors and lines in the projective plane as planes in $\mathbb{R}^3$, as the unique intersection of some oriplane with the projective plane.

However, if you'd prefer to skip all that, then my commentary on the Hestenes-Ziegler paper starts at Section 12.

2 Clifford algebra vs. Geometric Algebra

The mathematician William Kingdon Clifford (1845–1879) invented or reinvented what we call Clifford algebra as an extension of Grassmann algebra. Clifford himself called it 'geometric algebra'. In the decades after Clifford's death, his system ended up in relative obscurity, until its later matrix formulation version, which was used by physicists in quantum mechanics. Then onto the scene came David Hestenes, who singled handedly gave the subject a rebirth in a wide variety in physics topics. However, his connection to Clifford algebra was bridged by the wonderful book of Marcel Riesz that reformulated the classical equations of Maxwell into Clifford algebra. However, following the lead of Hestenes, many other physicists and mathematicians have pushed the development of applied geometric algebra, and recently with huge applications to computer science.

According to Hestenes, the label 'geometric algebra' should be restricted to a Clifford algebra in which the scalars are strictly real numbers. In a general Clifford algebra, the scalars can be any field. They are usually taken as the field of complex numbers.

Hestenes has done quite well in his decades-long research program to reformulate the foundations of physics by replacing matrices, quaternions, complex numbers, and scalar tensors by the appropriate counterparts in some geometric algebra. It doesn't take long in investigating geometric algebras to find an abundance of objects within the algebra that have squares of -1. Hence, Hestenes has added to his research program that there shall never be accepted any formal (or rather, uninterpreted in the geometric sense) square roots of -1. His philosophy has been that if one gets lazy and just finds one's square roots of minus one in the field of complex numbers, then one maybe missing a wonderful opportunity to find a deeper meaning to the square root of minus one in the form of some nonscalar element of the geometric algebra.

Now, with all that said about the distinction between Clifford algebra and geometric algebra, we shall not in this paper, or series of papers, make much distinction between them. So, why did I include this explanation? Because I know how confused I'd be if I were new to this subject but didn't know how they are related to the subject of projective geometry with Clifford algebra.

3 Introduction to this paper

At this time, I intend to go only about halfway through the Hestenes-Ziegler paper. That should provide the reader with some introduction to the entire paper and take the development of the subject using geometric algebra up to Pascal's Theorem, which is basically the Theorem of Desargues placed into an ellipse.

4 The unreasonable effectiveness of projective geometry

In my opinion, the 2-d projective plane is the most difficult and confusing 'simple' subject ever invented in the halls of mathematics. Let's take an example. Consider the following planar figure of the Pappus construction:

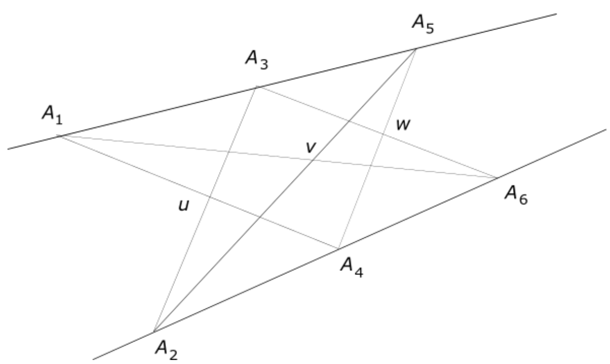

Figure 1. One version of Pappus's hexagonal planar figure. This representation hides that fact that to use projective geometry on it, one must embed the figure in a space of at least one dimension higher, which will be made clear later. But once you realize how the algebra of 3-d is used to prove theorems of figures in 2-d, the explicit representation of this embedding is not necessary, hence, the figures are displayed in 2-d to be easier on the eye, I suppose.

I'll state Pappus's theorem later, when we get to it in earnest. For now, it's obvious that it concerns a figure in a plane, or in 2-d. It's possible to prove this theorem in Euclidean geometry in 2-d. But, the form of projective geometry we're going to use in this paper doesn't do that. Projective geometry takes that plane, with figure inside, and places it inside a space of one dimension higher, so, in this case, in a 3-d space. Now, I'm going to call this 3-d space $\mathbb{R}^3$. The advantage of this space is that it is the well-known vector space that assigns coordinates to every point in the 3-d space. And I use the term 'vector space' quite specifically. In this vector space, we have a special point, called the *origin* which is the zero vector of the space, i.e., $\mathbf{0}$, such that for all vectors $\mathbf{v} \in \mathbb{R}^3$

$$\mathbf{v} + \mathbf{0} = \mathbf{v} \,. \tag{1}$$

Of course, the origin is given the coordinates $(0, 0, 0)$.

The magic happens when we embed our planar figure in $\mathbb{R}^3$ so that it doesn't contain the origin. There are two basic reasons for this. First, many of the proofs used in the projective plane use the vanishing of products of vectors (to get zero). Now, we want this resulting zero to indicate something specific about the relative geometry of the two or more vectors, not just the happenstance that one of the vectors is the zero vector, which gives us a zero, regardless.

Second, by placing the origin outside the plane of the figure, we are able to move from non-homogeneous coordinates to homogeneous coordinates.

I'm sure that projective geometry purists will complain that I am clouding the waters by embedding the projective plane into $\mathbb{R}^3$, but I have my reasons. My first reason is that so much of the literature out there uses coordinates to make proofs. All the proofs in Dorwart's book use coordinates. Next, every time you read the term 'homogeneous coordinates', you are dealing with coordinates of vectors and lines in $\mathbb{R}^3$.

By the way, the mystery of 'homogeneous coordinates' is explained below, and it's not all that hard to grasp. But it should be noted that in any coordinate-free proof, 'homogeneous coordinates' are not useful at all. And the proof afforded with the help of geometric algebra will not use coordinates at all.

Back to Figure 1. So, in the plane we have points and lines, and we are interested in the incidence relations among these objects. So, we're going to organize our treatment of projective geometry in three **Big Steps**.

Big Step I) In answer to the question of what are the objects of our concern in projective geometry, we shall at first refer to these objects in their simplest terms, as sets of points in $\mathbb{R}^3$. Of the unimaginably large number of subsets of points in $\mathbb{R}^3$, the only ones we will be interested in, in any precise sense are 1) single points, 2) lines, 3) planes, and 4) all of 3-d space. I shall refer to these four kinds of objects as *proper objects.* Except for the point, all other proper objects have infinite extent.[3]

Big Step IIa) The two main operations we will define on these proper objects are more or less set-theoretic operations of *meet* and *join.* Let A and B be any two proper objects in $\mathbb{R}^3$, then the join of these two objects, represented as $A \wedge B$, is the smallest proper object that setwise contains both A and B. For example, if A and B are both points in the projective plane, then their join is the unique line of the projective plane that contains them, or rather, that they both lie on.

Big Step IIb) Now for the meet. Let A and B be any two proper objects in $\mathbb{R}^3$, then the *meet* of these two objects, represented as $A \vee B$, is the largest proper object that is setwise contained in both A and B. Hence, the meet of two proper objects is very much their setwise intersections. An obvious first example is the meet of two lines. Lines are proper objects and their meet must

[3]It might be tempting to refer to some line segment in a particular planar figure, but such language is imprecise. In that case, what we are really dealing with is the line that contains the line segment.

be a proper object, and so it is. We say that, 'Two distinct coplanar lines *meet* at a point'. Likewise, two (distinct) planes meet in a line. Why is that? Because the line of intersection of the two planes is the largest subset of each plane that is contained in each plane. For contrast, any point in the intersection of the line of intersection of two planes is, of course, in the intersection of the two planes, but it is not the meet of the two lines.

Big Step IIc) So, why do we need both the meet and the join operations? We need them both to cover both notions of incidence in projective geometry. One notion of incidence is that of lines being incident at a point, say, and we have the meet operation for that. The other notion of incidence is that of points being incident on a common line, say, and we have the join operation for that. Desargues's famous theorem can be briefly stated as: When certain lines **meet** at a certain point, certain other points are **joined** on a certain other line. Although I just used highly specific examples to illustrate the meet and join operations, I need to repeat that the operations are to be defined on any two proper objects. The theorems of projective geometry of interest to us in this paper can be stated in terms of meets and joins of points and lines in a plane.[4]

Big Step IIIa) Okay, now that we know what the proper (or allowed) objects of discourse are, and we know the incidence concepts on these objects, and we have adopted symbols to represent these incidence relations on these proper objects, are we able to formulate an algebraic representation of these objects by which we can perform algebraic manipulations on them that can help us prove the theorems of projective geometry? The answer is yes. Dorwart and many other authors use the method of 'homogenous coordinates', in which all the points in space have coordinates like (a, b, c), and then operations of the Gibbs's vector algebra are used, even if they aren't described explicitly as such. Below, I prove the Pappus theorem by using Gibbs's vector algebra, but with no explicit adoption of coordinates to the vectors involved.

Big Step IIIb) After that, we turn to the real focus of this paper, which is how Hestenes and Ziegler used geometric algebra to represent these proper objects, so as to facilitate proving theorems with this mathematical system. Now, geometric algebra is superior to Gibbs's vector algebra in two main ways. First, it is an associative algebra that is generally easier to manipulate algebraically. Second, with respect to the theorems in the projective plane, we need to represent oriplanes[5] by some algebraic means. In the Gibbs's vector algebra, this is doable, but we are required to represent oriplanes by the cross products of two vectors in the plane.[6] However, though this is doable, it is not elegant.

[4]That the incidence theorems of projective geometry can be stated in terms of meets and joins, doesn't mean that they will be stated as such in the literature. For example, my book on projective geometry by H.S.M. Coxeter only mentions the word 'join' a few times and not as a formal operation. According to the book's index it doesn't mention 'meet' at all, but it mentions it on page 5, but not as a formal operation. Dorwart's book seems to make no formal use of them either.

[5]An oriplane is a plane in $\mathbb{R}^3$ that contains the origin.

[6]The oriplane needs some vector normal to it to represent it. This normal vector is usually derived from a cross product and two distinct vectors in the oriplane.

In the geometric algebra, we can represent a plane more directly by a bivector that shares the same 'direction' in space as does the plane. Secondly, we can easily generalize the geometric algebra to higher dimension, whereas, we cannot easily generalize the Gibbs's vector algebra to higher dimensions (because the cross product of vectors does not generalize easily), unless you want to think of geometric algebra as that generalization, which makes sense.

5 Introduction: Relating points in the projective plane to vectors in $\mathbb{R}^3$

The subject of projective geometry is vast. The purpose of this paper is to introduce only enough rudiments of the subject so as to get a feel for the subject and then to use this information to solve a real problem, namely, the Theorem of Pappus, also known as Pappus's hexagon theorem, and similar problems in later papers. More foundational proofs will be given in later papers when the topic of finite geometries comes up.

Now, in the abstract I promised to introduce homogenous coordinates, and I will, but only to connect their properties to Gibbs's vector algebra. What makes projective geometry work is a one-to-one correspondence in 3-space between planes and lines on the one hand and lines and points, respectively, on the other.

Here's the basic setup:

1) Take the planar figure you want to analyze (that naturally sits in $\mathbb{R}^2$) and embed it into $\mathbb{R}^3$ such that the origin $(0, 0, 0)$ is not in the plane. This structure is referred to as $\mathbb{RP}^2$. Let's choose a particular projective plane and call it Σ.

2) Note that there is a 1-1 correspondence between lines through the origin and points on Σ (i.e., where they intersect).

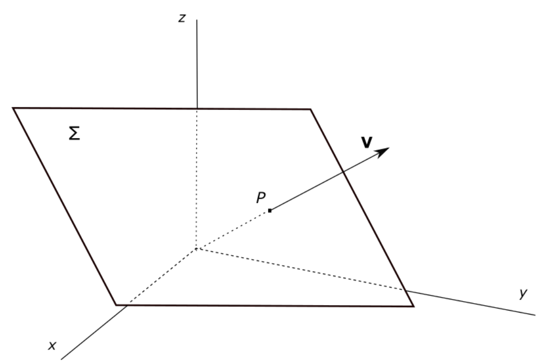

Figure 2. The 1-1 correspondence between lines through the origin, in this case represented by the vector $\mathbf{V}$, and points on the plane Σ. The intersection point of the line represented by the vector $\mathbf{V}$ and the plane Σ is the point P. What's important about the vector is not its length, but its direction in 3-space.

Now, it turns out that for projective geometry we don't need to describe the points in Σ as unique vectors from the origin to Σ. All that is needed is to find any vector along the line connecting the origin and the point in Σ. This will be proved later on.

Also note that there is a 1-1 correspondence between planes that contain the origin (which I refer to as *oriplanes*[7]) and lines in Σ. Thus, to test if three points in Σ are collinear, all we have to do is prove that the three points each correspond to three lines in some oriplane, which is easy to do with Gibbs's vector algebra. Why is this? Because, if the three points lie simultaneously on two distinct planes, they must lie on the intersection of the two planes, which is a line.

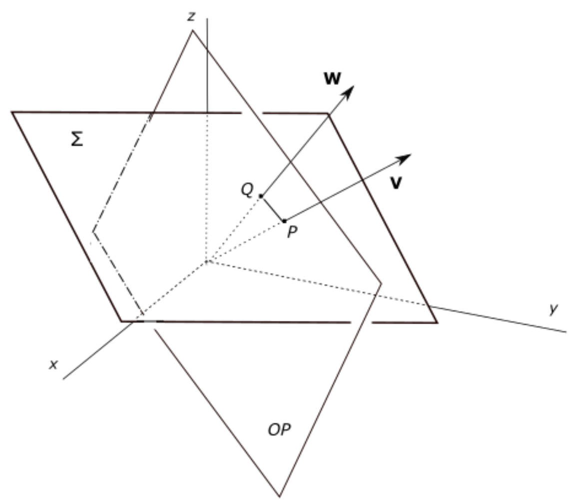

Figure 3. The 1-1 correspondence between lines through the origin, in this case represented by vectors $\mathbf{W}$, $\mathbf{V}$, and points Q, P, respectively, on the (projective) plane Σ. The oriplane OP is the unique plane through the origin containing the 'lines' $\mathbf{W}$ and $\mathbf{V}$. The orientation in 3-space of the plane OP can be represented by the normal vector $\mathbf{N}$ to the plane OP, given by $\mathbf{N} = \mathbf{V} \times \mathbf{W}$. What's important about this vector is not its length, but its direction in 3-space.

So, what is the secret to this mysterious replacement of lines in the plane $\mathbb{R}^2$ with homogeneous coordinates in $\mathbb{R}^3$? Simply this: The generic line in $\mathbb{R}^2$ with standard (non-homogenous) form

$$Ax + By + C = 0\,, \tag{2}$$

has been replaced by the line in $\mathbb{RP}^2$ Σ, with generic homogeneous form

$$Ax + By + Cz = 0\,. \tag{3}$$

[7]Of course, our oriplanes have no resemblance to the Japanese origami planes.

So, how is this done? Imagine we have an oriplane OP with its normal vector $\mathbf{N}$ having its base point at the origin. Imagine also that we have any other nonzero vector, say $\mathbf{X}$, in the same plane with its base point at the origin. Then the dot product of these two vectors is zero, for they are orthogonal to each other, i.e.,

$$\mathbf{N} \cdot \mathbf{X} = 0\,. \tag{4}$$

Now, imagine parallel transport of $\mathbf{X}$ to any other place in OP, and call it $\mathbf{X}'$. The wonderful thing is that by parallel transporting $\mathbf{X}$ from the origin, we have to move its base point and its end point by the same amount, call it $\mathbf{D}$. Therefore, with $\mathbf{0} = (0, 0, 0)$ and $\mathbf{X} = \mathbf{X} - \mathbf{0}$,

$$\mathbf{X}' = (\mathbf{X} + \mathbf{D}) - (\mathbf{0} + \mathbf{D})\,, \tag{5}$$

where, because of the translation away from the base point at the origin, $\mathbf{X}'$ is no longer a 'proper' vector with coordinates, but is now an affine vector with components. By a 'proper' vector I mean a vector whose base point is always at the origin. An affine vector can have its base point anywhere. So, if we dot this affine vector by $\mathbf{N}$, we get

$$\begin{aligned} \mathbf{N} \cdot \mathbf{X}' &= \mathbf{N} \cdot (\mathbf{X} + \mathbf{D}) - \mathbf{N} \cdot (\mathbf{0} + \mathbf{D}) \\ &= \mathbf{N} \cdot \mathbf{X} + \mathbf{N} \cdot \mathbf{D} - \mathbf{N} \cdot \mathbf{0} - \mathbf{N} \cdot \mathbf{D} \\ &= \mathbf{N} \cdot \mathbf{X} = 0\,. \end{aligned} \tag{6}$$

So, what we've shown is that the normal to a plane is orthogonal to every vector in the plane, and to every line in the plane, and to every line segment in the plane.

Going back to Figure 3, let us regard the vector $\mathbf{X}$ as the vector going from point P to point Q, and let it have components

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\,. \tag{7}$$

Now let $\mathbf{N}$ be the normal to oriplane OP,[8] having coordinates (or components, since it doesn't matter in this case)

$$\mathbf{N} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}\,. \tag{8}$$

The components of vector $\mathbf{X}$ have, so far, only one constraint on them, given by Eq. (4), therefore,

$$\mathbf{N} \cdot \mathbf{x} = [A, B, C] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\,, \tag{9}$$

[8] Actually, we can rescale this normal by any nonzero factor and, for the purposes of projective geometry, it's still 'the' normal.

which gives us Eq. (3). Thus, Eq. (3) represents a line in the oriplane OP that goes through the origin and is parallel to line segment $\overline{QP}$.

We're now at the point where we can set down the two building blocks of incidence geometry from the projective geometry viewpoint:

1) Given three points in the projective plane, do we have a simple test to determine if they are incident with one line?

2) Given that all distinct pairs of lines in the projective plane intersect in a point, do we have a means of representing that intersection point in terms of the points on the two lines?

We'll deal with the first question now.

Figure 4 is an extension of Figure 3 by adding the named point R to Σ. It looks like it lies on the same line that contains points P and Q, but we need a test for this. These three points are collinear if and only if they lie on the line of intersection of planes OP and Σ.

Now we make a critical simplification. We're going to treat the points P, Q, and R as the tips of vectors in $\mathbb{R}^3$ with base points at the origin $\mathbf{0}$.

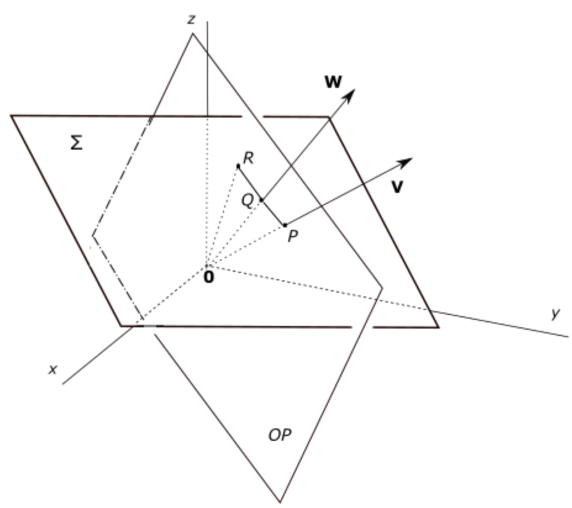

Figure 4. Compared to Figure 3, we've added the point R in the projective plane Σ. We look for a means to determine algebraically if the three points P, Q, and R are collinear.

If points P, Q, and R are collinear, then vectors P, Q, and R lie in the same oriplane, namely OP, in which case, every one of those vectors is orthogonal to any normal to the plane. We can find a normal vector to OP by taking the cross product of any two of those three vectors, such as $R \times Q$ or $R \times P$ or $P \times Q$, etc.[9] Then we have that

$$P \cdot (R \times Q) = Q \cdot (R \times P) = R \cdot (P \times Q) = 0\,. \tag{10}$$

It may seem at this point of the discussion a bit odd not to use boldface for vectors, such as

$$\mathbf{P} \cdot (\mathbf{R} \times \mathbf{Q}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \mathbf{R} \cdot (\mathbf{P} \times \mathbf{Q}) = 0\,, \tag{11}$$

but it's best to learn to do without that affect because the standard literature does not use it. Hereafter, in this first paper, I'll use it at times.

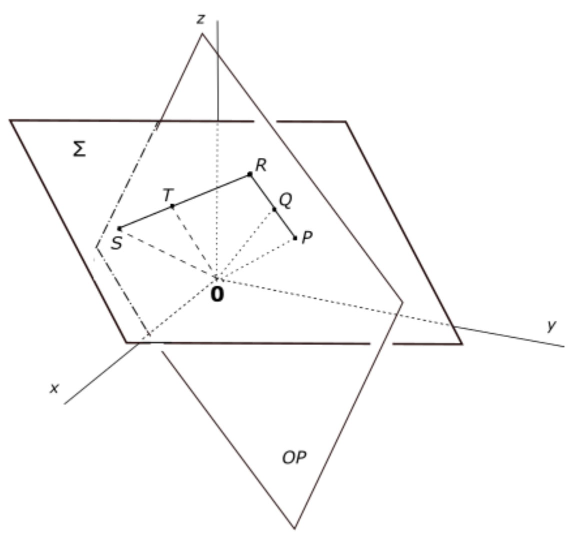

Figure 5. Compared to Figure 4, we've added the points S and T in the projective plane Σ, which lie in a different oriplane than OP, which we'll call OP' (not shown to reduce clutter). Lines ST and PQ will meet at point R.

Now, how do we find the point of intersection of two distinct lines in the projective plane Σ? Refer now to Figure 4. Let R be the point of intersection of line $\overline{PQ}$ and $\overline{ST}$. But how do we represent point R algebraically in terms of the given information of points P, Q, S, and T? Well, this being projective geometry, not Euclidean geometry, we are only concerned with determining the direction of R in $\mathbb{R}^3$. For that limited purpose, all we need is any vector on the line $\overline{\mathbf{0}R}$.

The key to finding the algebraic expression that will give us a representation of the direction of the point R is first to realize that the line $\overline{\mathbf{0}R}$ is along the

[9] We can also take these cross products in their opposite orders, but introducing a minus sign won't matter, since we are trying to get an inner product of zero, anyway. And minus zero is the same as plus zero.

line of intersection of the two oriplanes OP and OP', and second, to realize that we can find that direction from taking the cross product of the normals to those two oriplanes. (The line of intersection of two planes is orthogonal to the normals of both planes, hence, the cross product of those normals.) Thus, the 'projective location' of the point R is given by

$$R \sim (P \times Q) \times (S \times T)\,. \tag{12}$$

If we were to write this equation in terms of 3-space vectors, we would get

$$\mathbf{R} = \lambda(\mathbf{P} \times \mathbf{Q}) \times (\mathbf{S} \times \mathbf{T})\,, \tag{13}$$

where λ is some unknown nonzero real number. However, we don't need to know the value of λ to answer the kind of questions we will encounter in incidence geometry in this series of papers. Hence, we will confidently write the 'projective location'[10] of the point R as given by (effectively setting λ to unity), thus

$$R = (P \times Q) \times (S \times T)\,. \tag{14}$$

One last point to make in this section. There is a unique oriplane in $\mathbb{R}^3$ that is parallel to Σ in the Euclidean sense. Call this plane the *ideal* plane. However, we will not make much use of this plane in this series of articles.

6 Last detours before Pappas's Hexagonal theorem

Before proving Pappas's Hexagonal theorem, we need 1) to add a few definitions for use in the series and to make contact with the literature; 2) to review (briefly) those few aspects of inner products and determinants that we'll use repeatedly in this series to make actual calculations to solve our problems; and 3) to introduce the compact 'bracket' notation. This and the next few sections will accomplish this task. Let's begin.

Definition 1:
Projective Geometry is the area of mathematics concerned with the invariants of figures under projective transformations, in particular, central projections.

So, what are these invariants of figures under projections? First, what are the figures? The figures, for our restricted purposes, are planar figures of lines and points. The invariants are 1) the indicated points in the given plane, 2) the lines, and 3) the incidence relations between lines and between points on lines.

[10] I coined this term.

Thus, if two lines are incident in the figure, they are incident in the projection. For our purposes, we consider the projections to be between one plane sitting in 3-space to another plane in 3-space, or a sequence of such projections.

Definition 2:
A collineation is a bijection from one projective space to another that preserves incidences.

Definition 3:
Two distinct lines *meet* (symbolically $\vee$) at a point, and two distinct points are joined by (that is, lie on) a line (symbolically $\wedge$). Thus, if ℓ_1 and ℓ_2 are two lines that meet at point p, we can write: $p = \ell_1 \vee \ell_2$. And if p_1 and p_2 are two points that lie on line ℓ, we can write: $\ell = p_1 \wedge p_2$. Clearly, once ℓ is determined, any two distinct points on that line will have meet equal to ℓ.

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Projective geometry comes in two flavors: Synthetic and Analytic. In the synthetic version, the subject begins with a statement of the axioms the geometry obeys, and it leave the terms *point* and *line* as undefined (primitive) terms. It also doesn't use coordinates, but, rather, proceeds similar to how synthetic Euclidean geometry is founded on the existence of points and lines and incidences, etc.

Analytic Projective geometry begins typically with $\mathbb{F}^3$, where $\mathbb{F}$ is a field. Our field of choice in this paper is the real numbers.

Besides a knowledge of some linear algebra and Gibbs's vector algebra, the rest of the mathematical preparation the reader will need to follow the proof of Pappus's hexagonal theorem is begun in Chapter 4.

7 Homogeneous Coordinates

First, a little history. The ancient Greeks knew that the conic figures can be obtained by intersecting a plane with a cone. Later, with the idea of projections came the realization that distances and angles are generally not preserved. What is preserved, then? Incidence relationships are. Any two distinct points in a plane determine a line. But the relationship between lines is not so perfect: Any two nonparallel lines in a plane meet at a single point. However, in Euclidean geometry, parallel lines never meet. This seems like a sort of spontaneously broken symmetry between points and lines in a plane.

If we could fix this defect somehow, then there would be a perfect symmetry between theorems about incidence relationships about points and lines in a plane. In other words, to force a duality on the points and lines in a plane so that all theorems about them remain true when we interchange the words 'point' and 'line' is to adopt the convention that all parallel lines in a given direction meet at a single point infinity.

Now imagine having a plane in R^3 with z value unity. That is, the plane $z =$

1. Every line in that plane can be expressed as the set of points of intersection of the $z = 1$ plane with some plane through the origin $(0, 0, 0)$. The general equation that describes such a plane (i.e., an oriplane) is

$$a_1 x + a_2 y + a_3 z = 0\,, \tag{15}$$

where a_1, a_2, a_3 are real numbers. This equation is said to be *homogeneous* because when all its constant terms are collected into one lumped term, that term is zero, which is true if and only if the plane contains the origin.

At this point in the development, we change gears completely by adopting the methods and notations of Gibbs's vector algebra. (Since this paper assumes the reader is already familiar with this algebra, we won't go into a deep presentation of it here.) In this algebra, a vector is a directed line segment from a base point to its tip. The angle between any two vectors can be established by calculating the dot or inner product between the vectors.

Say we have two free nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in 3-space, and we want to find the angle between them. How do we do this? We bring their base points together (at any convenient point in space, since it doesn't matter) and then compute their dot product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta\,, \tag{16}$$

where θ is the angle between the two vectors. When this angle is $\pi/2$, the cosine of the angle is zero and (16) becomes

$$\mathbf{a} \cdot \mathbf{b} = 0\,, \tag{17}$$

in which case $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.

We can express Equation (15) in vector form this way

$$\mathbf{a} \cdot \mathbf{x} = 0\,, \tag{18}$$

where $\mathbf{x}$ is a vector (with base point at the origin) perpendicular to the vector $\mathbf{a}$, which also has its base point at the origin. The locus of all points $\mathbf{x}$ in $\mathbb{R}^3$ defined by (18) is the oriplane perpendicular to the vector $\mathbf{a}$.[11]

The whole point of homogeneous coordinates is to enforce a one-to-one correspondence between the points on the projective plane and the vectors that start at the origin and end at the points on the plane. This is why I refer to points as vectors and vice versa.

There are two ways to proceed at this point. One way is to explicitly adopt coordinates to prove theorems in projective geometry by choosing ones easy for computations, such as $(0, 0, 1)$ or $(0, 1, 0)$, etc. We lose nothing in generality by choosing such coordinates because all Pappus figures are projectively equivalent.[12] And now it is obvious why we choose the projective plane to have z coordinate unity.

[11] To ensure that the origin is included in this locus of points, giving us the entire oriplane, we must insist that the zero vector is orthogonal to every vector.

[12] In this concept I include the great freedom allowed to draw the figure in the first place.

The other way to proceed is to leave coordinates as implicit and just use the power of dot and cross products on arbitrary points in the projective plane, as is the procedure in this paper. On the other hand, Dorwart used coordinates for his proofs, whereas I'll use the noncoordinate method. Either way, it's still just the Gibbs's vector algebra adapted for use in projective geometry.

Two questions might occur to the reader at this point. The first is, Why we need the projective plane to have z coordinate unity if we're going to ignore the explicit use of coordinates in the latter way to do projective geometry? The second question is, Why can't we leave the origin in the projective plane, rather than embed the plane into a larger space?

In answer to the first question, If we aren't going to assign particular coordinates to points in the projective plane, there really is no need to use the canonical construction of the plane with z coordinate equal to 1. In this case, all we need is that the projective plane not include the origin, or put another way, that the origin is not in the projective plane. And *that* is why projective geometry adds another spatial dimension to the problem.

But surely there are proofs of Pappus's theorem that only use the 2-D plane and not 3-space, right? Right. I've seen proofs using Euclidean geometry, and I invented a proof using isotropic spinors in the plane. I can easily imagine a proof using geometric algebra in the plane. So why do we embed the Pappus planar figure in 3-space? You might say, We do so, so we can get a better perspective on the problem. And, humor, aside, this is quite literally the case.

One reason to embed the projective plane into a space of higher dimension is to deal more effectively with those nasty 'points at infinity', and I'll get to that issue later. The other reason we do this is to avoid irksome special conditions that we'd have to be watchful for if the origin were inside the plane, and which are never a problem if the origin is outside the plane. And this is a nice segue that leads us to a full presentation of the technical aspects of how to represent *line intersection* and *collinearity* in terms of the Gibbs's vector algebra.

8 Technical Preliminaries

Let's get technical with a little vector algebra and linear algebra now in $\mathbb{R}^3$, as well as a review of the previous sections. In the projective plane every pair of distinct lines intersect, or *meet*, at a unique point. If the lines are parallel, they meet at a point at infinity. Otherwise, they meet at a finite point, just like in Euclidean geometry. Every line in the projective plane is the intersection of a unique plane through the origin and the projective plane. Henceforth, we'll refer to the projective plane as Σ.

Every plane through the origin can be uniquely defined (up to a sign) by its unit normal vector, call it $\mathbf{n}$, where $\mathbf{n} \cdot \mathbf{n} = 1$. The equation that defines the locus of points $\mathbf{x}$ in space that are in the plane perpendicular to $\mathbf{n}$ is given by

$$\mathbf{n} \cdot \mathbf{x} = 0\,. \tag{19}$$

However, because (19) is homogeneous, we can be more general in our characterization of this plane by an arbitrary nonzero rescaling of $\mathbf{n}$. So, let λ be any nonzero real number, and let $\mathbf{a} = \lambda\mathbf{n}$, then by multiplying (19) through by λ, the equation can be equivalently recast as

$$\mathbf{a} \cdot \mathbf{x} = 0\,. \tag{20}$$

Now, so long as this vector $\mathbf{a}$ is not normal to Σ, the plane normal to $\mathbf{a}$ will intersect Σ in the Euclidean sense. So, say there are two distinct lines in Σ, L_1 and L_2, say, that intersect at point p. Then there are two distinct plane Σ_1 and Σ_2 through the origin that intersect Σ in lines L_1 and L_2, respectively. (Consult Figure 5.)

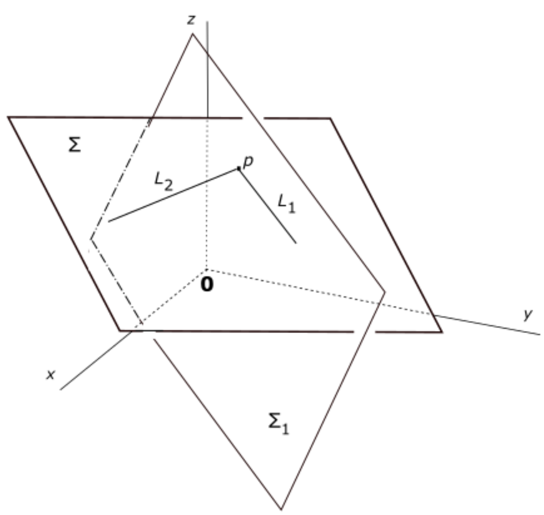

Figure 6. Line L_1 in oriplane Σ_1 meets line L_2 in oriplane Σ_2 (not shown to reduce clutter) at point p.

Let Σ_1, Σ_2, and Σ, be given by, respectively,

$$a_1x + a_2y + a_3z = 0\,, \tag{21a}$$
$$b_1x + b_2y + b_3z = 0\,, \tag{21b}$$
$$c_1x + c_2y + c_3z = d\,, \tag{21c}$$

where d and all the coefficients are real numbers, and in particular, d is nonzero. Or, expressed in matrix form

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}\,. \tag{22}$$

The simultaneous solution to these equations gives us the exact point $\hat{p}$ in Σ, namely,

$$\hat{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \frac{d}{\det M} \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} , \tag{23}$$

where M is the coefficient matrix in (22), and its determinant is nonzero:

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} . \tag{24}$$

Now, we don't need the exact point $\hat{p}$ in Σ. We only need the direction of $\hat{p}$ to characterize it for our purposes. Any vector along the ray defined by $\hat{p}$ will do. To that end, we set

$$\mathbf{p} = \begin{bmatrix} b_1 c_2 - b_2 c_1 \\ a_2 c_1 - a_1 c_2 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} , \tag{25}$$

where we have ignored nonzero factors of this vector that appeared in (23).

The reader might suspect that, just as the vector[13] $\mathbf{a} = [a_1, a_2, a_3]^{\mathrm{T}}$ is perpendicular to Σ_1 and vector $\mathbf{b} = [b_1, b_2, b_3]^{\mathrm{T}}$ is perpendicular to Σ_2, that vector vector $\mathbf{c} = [c_1, c_2, c_3]^{\mathrm{T}}$ is perpendicular to Σ, and this is correct. Let's prove it.

Let $\mathbf{X} = [X_1, X_2, X_3]^{\mathrm{T}}$ be some particular point that satisfies (21c), that is, is a point on Σ, and let $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ be any other point on Σ. Now, since $\mathbf{X}$ and $\mathbf{x}$ are both points in Σ then their difference is a vector lying in the plane Σ. So, substitute $\mathbf{X}$ and $\mathbf{x}$ into (21c) and subtract the resulting two equations to get

$$c_1(X_1 - x_1) + c_2(Y_2 - y_2) + c_3(X_3 - x_3) = 0 \, . \tag{26}$$

But this equation can be rewritten as

$$\mathbf{c} \cdot (\mathbf{X} - \mathbf{x}) = 0 \, . \tag{27}$$

But since $\mathbf{X} - \mathbf{x}$ is an arbitrary vector in Σ and $\mathbf{c}$ is normal to it, then $\mathbf{c}$ is normal to Σ.

A useful feature of the determinant of a square matrix is that the effect of swapping any two rows of the matrix will only multiply the resulting determinant of the new matrix by -1. Therefore, any sequential even number of such swaps with leave the value of the resulting determinant unchanged, and any odd number will multiply it by -1. A cyclic permutation of the rows of a 3×3 matrix is an even number of swaps and so will not change the sign of the determinant.

Let's cyclically permute the rows of M in (24), bringing the bottom row to the top:

$$M' = \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} , \tag{28}$$

[13] The superscript T means to take the matrix transpose.

and the determinant of M' is

$$\det M' = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} . \tag{29}$$

Normally, the determinant is scalar-valued, but Gibbs's algebra allows us to write vector-valued determinants, like this one

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \,, \end{aligned} \tag{30}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of the x, y, z-axes, respectively. Of course, $\mathbf{a} \times \mathbf{b}$ is the cross product of $\mathbf{a}$ and $\mathbf{b}$, and is a vector orthogonal to both of these vectors. Now, if we dot this through by some arbitrary vector $\mathbf{c}$, we get

$$\begin{aligned} \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3 \,. \end{aligned} \tag{31}$$

To resolve any ambiguity here, the expression $\mathbf{c} \cdot \mathbf{a} \times \mathbf{b}$ is always to be interpreted as $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

By straightforward calculations using (31), one can prove that

$$\mathbf{a} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \times \mathbf{b} = 0 \,. \tag{32}$$

This last result is also known from linear algebra, which proves that the determinant of any nontrivial $n \times n$ matrix is zero if any two rows have the same components.

Lemma:

$$\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \,. \tag{33}$$

We now have an interpretation of (29) in terms of the dot and cross products of vectors built out of the rows of the matrix M'. We can generalize our claim from linear algebra about when the determinant of a square matrix is zero. A determinant is zero if and only if its rows are linearly dependent.[14] A vector is linearly dependent on a set of other vectors, if it can be written as a linear combination of the other vectors.

[14]We won't have to concern ourselves with the case when one of the rows is the zero vector, because the zero vector (i.e., the origin) is not a point of the projective plane—by design.

In the case of the three vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, say that $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$. We'll view this linear dependence both algebraically and geometrically. First, algebraically. Then we can write for some nonzero scalars α and β

$$\mathbf{c} = \alpha\mathbf{b} + \beta\mathbf{c}\,. \tag{34}$$

Then, applying (32),

$$\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = (\alpha\mathbf{a} + \beta\mathbf{b}) \cdot \mathbf{a} \times \mathbf{b} = \alpha\mathbf{a} \cdot \mathbf{a} \times \mathbf{b} + \beta\mathbf{b} \cdot \mathbf{a} \times \mathbf{b} = 0\,. \tag{35}$$

Viewed geometrically, this means that $\mathbf{c}$ is in the same plane as determined by $\mathbf{a}$ and $\mathbf{b}$. Then, since $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane containing $\mathbf{a}$ and $\mathbf{b}$, it is orthogonal to $\mathbf{c}$ as well. That is, $\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = 0$.

9 Line Intersection and Collinearity

Let $\mathbf{a}$ and $\mathbf{b}$ be distinct points in Σ. Then the plane through the origin that contains these points intersects Σ in a line that contains points $\mathbf{a}$ and $\mathbf{b}$. A vector normal to this plane is the vector $\mathbf{a} \times \mathbf{b}$. Thus the join of points $\mathbf{a}$ and $\mathbf{b}$ in Σ, that is, the line ℓ containing them, is characterized briefly by $\mathbf{a} \times \mathbf{b}$. The set of points on ℓ is the locus of points $\mathbf{x}$ in Σ orthogonal to $\mathbf{a} \times \mathbf{b}$. In other words, $\mathbf{x}$ is 'on the line' ℓ if and only if

$$\mathbf{x} \cdot \mathbf{a} \times \mathbf{b} = 0\,, \tag{36}$$

in which case we say that points $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ are *collinear* in the projective sense.

"Now wait a minute!" I hear you say. "Didn't you claim recently that an entire oriplane orthogonal to the vector $\mathbf{a} \times \mathbf{b}$ is defined by Equation (36), not just a point on a line?" To which I reply that that's right, when regarding (36) as an equation in $\mathbb{R}^3$. But now consider a ray from the origin to infinity in that oriplane. It must intersect the line ℓ in some point. However, considered projectively, every point on that ray is equivalent to that point of intersection because they all share one critical distinction in $\mathbb{R}^3$, namely, they all share the same direction in 3-space.

Definition: The *triple scalar product* of the three arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ shall be given by $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

Now for the join of two lines in the projective plane. We've already seen that a line in Σ is defined by two distinct points in Σ, and since the meet of two lines in the projective plane is a point, we need four distinct points, two for each line, to define this meet of the lines. Let $\mathbf{x}$ be this unique point of intersection of these two lines, $\ell_1 = \mathbf{a} \times \mathbf{b}$ and $\ell_2 = \mathbf{c} \times \mathbf{d}$. Since $\mathbf{x}$ lies on each line, it must satisfy the two equations

$$\mathbf{x} \cdot \mathbf{a} \times \mathbf{b} = 0 \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{c} \times \mathbf{d} = 0\,. \tag{37}$$

There is an obvious choice for the solution to this couple of constraints, and that is

$$\mathbf{x} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})\,. \tag{38}$$

This solution for $\mathbf{x}$ is unique, up to arbitrary nonzero scale factor, because two distinct lines can only intersect on a single point. And as a reminder, $\mathbf{x}$ as computed by Equation (38) is said to be the 'projective location' of the point $\mathbf{x}$.

However, the triple cross product is not too convenient for calculations for my tastes. Let's employ the double cross product vector identity to reexpress it more simply:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\mathbf{A} \cdot \mathbf{C} - \mathbf{C}\mathbf{A} \cdot \mathbf{B}\,, \tag{39}$$

a result familiar to most calculus students. So, start with

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = -[(\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{b})]\,, \tag{40}$$

and then set

$$\mathbf{A} \to \mathbf{c} \times \mathbf{d}, \quad \mathbf{B} \to \mathbf{a}, \quad \mathbf{C} \to \mathbf{b}\,, \tag{41}$$

and then, using (39), (40) continues on with

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= -[(\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{b})] \\
&= -[\mathbf{a}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b} - \mathbf{b}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}] \\
&= \mathbf{b}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a} - \mathbf{a}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b} \\
&= \mathbf{b}\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}) - \mathbf{a}\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})\,. \qquad \text{(42a)}
\end{aligned}$$

However, vectors $\mathbf{a}$ and $\mathbf{b}$ don't get all the glory. By a similar maneuver, we can write

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{c}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) - \mathbf{d}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\,. \tag{42b}$$

Hint:

$$\mathbf{A} \to \mathbf{a} \times \mathbf{b}, \quad \mathbf{B} \to \mathbf{c}, \quad \mathbf{C} \to \mathbf{d}\,, \tag{43}$$

and then use cyclic permuting of the vectors in the triple scalar products.

10 The Compact Bracket Notation

Convention in this subject employs a 'bracket' notation to make the equations in projective geometry much easier write down and to comprehend. I wholeheartedly endorse this practice. Let's look at my take on them.

Look at Equations (42a) and (42b). They're just vector equations in the Gibbs's algebra. But if we include every dot and cross product, the expressions and equations would be full of distracting symbols that really aren't needed. The proof that they're not needed is that we can define a bracket notation that unambiguously does away with them.

So how does this bracket notation work? So far, I have found only four bracket types to use in proving theorems in projective geometry. They are

$$[\mathbf{a}], \quad [\mathbf{ab}], \quad [\mathbf{abc}], \quad [\mathbf{abcd}]\,, \tag{44}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are vectors/points in the projective plane. As is true in the Gibbs's vector algebra, all these expressions represent either scalars or vectors, though vectors have two different interpretations in projective geometry: The following three expressions

$$[\mathbf{a}], \quad [\mathbf{ab}], \quad [\mathbf{abcd}]\,, \tag{45}$$

are vector-valued, as given by the following definitions:

$$[\mathbf{a}] \equiv \text{the point } \mathbf{a} \text{ in the projective plane,} \tag{46a}$$

$$[\mathbf{ab}] \equiv \mathbf{a} \times \mathbf{b} \text{ represents a line in the projective plane,} \tag{46b}$$

$$[\mathbf{abcd}] \equiv (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) \text{ represents a point in the projective plane.} \tag{46c}$$

In terms of the jargon already laid out, $[\mathbf{ab}]$ is the join of points $\mathbf{a}$ and $\mathbf{b}$ (i.e., the line containing $\mathbf{a}$ and $\mathbf{b}$). And $[\mathbf{abcd}]$ is the meet of lines $[\mathbf{ab}]$ and $[\mathbf{cd}]$. Whereas,

$$[\mathbf{abc}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}\,, \tag{47}$$

which, as a reminder, is the *triple scalar product*. We've already seen this quantity to represent a determinant, with the properties

$$[\mathbf{abc}] = [\mathbf{cab}] = [\mathbf{bca}] = -[\mathbf{bac}] = \text{ etc}\,. \tag{48}$$

Of course, if any one of the triple scalar products in (48) is zero, they all are. And the scalar product of three nonzero vectors is zero if and only if they're coplanar.

One final obvious point to make is that, based on (32), for all $\mathbf{a}, \mathbf{b}$

$$[\mathbf{aab}] = [\mathbf{bab}] = 0\,. \tag{49}$$

In the Pappus theorem, points in the figure are labeled with subscripts. I could represent the point A_1, for example, as $[A_1]$, but, in the interest of minimality, I should go all the way and just write [1] for it. And this is allowable so long as it doesn't introduce an ambiguity.

I can also mix subscripted with unsubscripted points in a bracket. For example, to claim that the three points A_2, u, and A_3 are collinear, all I need write is[15]

$$[2u3] = 0\,, \tag{50}$$

which, of course, stands for

$$A_2 \cdot u \times A_3 = 0\,. \tag{51}$$

[15] For the time being, I will treat boldface and non-boldface vectors the same. Perhaps later I'll find a reason to distinguish them.

Equation (39) for the double cross product can be recast as

$$
\begin{aligned}
[\mathbf{A}] \times [\mathbf{BC}] &= [\mathbf{B}]\mathbf{A} \cdot \mathbf{C} - [\mathbf{C}]\mathbf{A} \cdot \mathbf{B} \\
&= [\mathbf{B}][\mathbf{A}] \cdot [\mathbf{C}] - [\mathbf{C}][\mathbf{A}] \cdot [\mathbf{B}]\,. \qquad (52)
\end{aligned}
$$

Lastly, we look at some identities involving the triple cross product $[abcd]$. Remember that this is the meet point of the two lines $[ab]$ and $[cd]$. In our bracket notation, (40) becomes

$$[abcd] = -[cdab]\,. \qquad (53)$$

Note that $[abcd]$ also changes sign if we transpose either a and b, or c and d. And (42a) and (42b) become

$$
\begin{aligned}
[abcd] &= [b][acd] - [a][bcd] \qquad (54a)\\
&= [c][abd] - [d][abc]\,. \qquad (54b)
\end{aligned}
$$

EXERCISE: An Identity

Start with $[abcd] = [b][acd] - [a][bcd]$ and take the cross product of it on both sides of the equation on the left with vector $e = [e]$, to get[16]

$$[ab][ecd] - [cd][eab] = [eb][acd] - [ea][bcd]\,. \qquad (55a)$$

Now, dot (55a) with vector $c = [c]$ on both sides to get

$$[cab][ecd] = [ceb][acd] - [cea][bcd]\,. \qquad (55b)$$

Lemma 1 (on the meet of distinct lines)
Let A, B, C, D be distinct points in the projective plane. Then

$$
\begin{aligned}
[ABCD] &= B[ACD] - A[BCD] \qquad (56a)\\
&= C[ABD] - D[ABC]\,. \qquad (56b)
\end{aligned}
$$

Remember that $[ABCD]$ is a vector/point and is equal to $(A \times B) \times (C \times D)$

Lemma 2 (on the collinearity of points)
Three distinct points A, B, C in the projective plane are collinear if and only if $[ABC] = 0$.

More Practice

Let's do some examples scalarizing vectors. Say we have the vector equation

$$A = \lambda B + C\,, \qquad (57a)$$

[16]Hint: Remember that $[abcd]$ is a cross product $[ab] \times [cd]$, so that $[e] \times [abcd] = [e] \times ([ab] \times [cd])$, so use the double cross product formula (52).

where A, B, C are vectors and λ is a scalar. If we cross (57a) by, say, the vector D on the right, we get

$$A \times D = \lambda B \times D + C \times D\,. \tag{57b}$$

Now we can dot this last equation on the left by, say, the vector E to get

$$E \cdot A \times D = \lambda E \cdot B \times D + E \cdot C \times D\,, \tag{57c}$$

where, again, the expression $A \cdot B \times C$ is always to be interpreted as $A \cdot (B \times C)$. And, of course, we want to rewrite the last three equations into our bracket notation, yielding

$$[A] = \lambda[B] + [C]\,, \tag{58a}$$

$$[AD] = \lambda[BD] + [CD]\,, \tag{58b}$$

$$[EAD] = \lambda[EBD] + [ECD]\,. \tag{58c}$$

To demonstrate dotting a vector equation with a cross product, consider dotting (57a) with $D \times E$ to get

$$A \cdot D \times E = \lambda B \cdot D \times E + C \cdot D \times E\,, \tag{59a}$$

or, in bracket notation,

$$[ADE] = \lambda[BDE] + [CDE]\,. \tag{59b}$$

A shorthand way to think about how we went from (58a) to (58c), was to apply the operation of $[E\llcorner D]$ across (58a). And the alternative way to scalarize a vector equation is to apply the following cross product $[\llcorner DE]$, such as in going from (58a) to (59b).

11 Theorem of Pappus

So far as I can tell, Pappus merely stated the following theorem, but never proved it:

As shown in Figure 6, let there be two distinct lines in a plane, each having three distinct points on them, A_1, A_3, A_5 on one, and A_2, A_4, A_6 on the other. (A particular labeling of these points is quite arbitrary.) Then, the points of the intersections (i.e., the meets) of the given pairs of interior lines (i.e., joins) are collinear. That is, points u, v, w are on the same line.

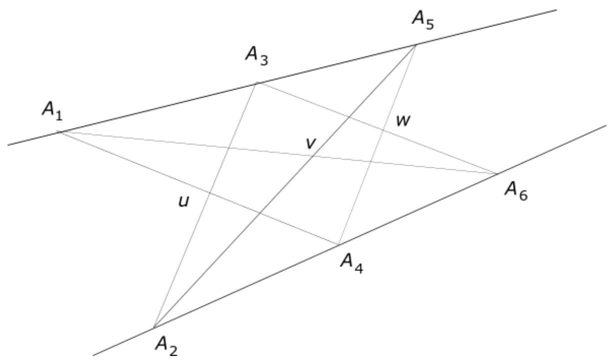

Figure 7. One version of Pappus's hexagonal planar figure.

Proof of Pappus's Theorem: STEP 1 TO THE PROOF: Collect the constraints.

As presented in the figure, $u = \overline{A_1A_4} \vee \overline{A_2A_3}$, $v = \overline{A_1A_6} \vee \overline{A_5A_2}$, and $w = \overline{A_3A_6} \vee \overline{A_5A_4}$, or, presented in our compact notation

$$u = [1423]\,, \tag{60a}$$
$$v = [1652]\,, \tag{60b}$$
$$w = [3654]\,. \tag{60c}$$

Thus, we can recast the essence of the claim of the theorem in this form: Show that

$$[uvw] = 0\,, \tag{61}$$

given that 1)

$$[135] = 0\,, \tag{62a}$$
$$[246] = 0\,, \tag{62b}$$

and given that 2)

$$[1u4] = 0\,, \tag{63a}$$
$$[3u2] = 0\,, \tag{63b}$$
$$[1v6] = 0\,, \tag{63c}$$
$$[5v2] = 0\,, \tag{63d}$$
$$[3w6] = 0\,, \tag{63e}$$
$$[5w4] = 0\,. \tag{63f}$$

STEP 2 TO THE PROOF: In order to show that $[uvw] = 0$, find expressions for u, v, w in terms of the given points.[17]

Thus, using (56a), we can solve for $A_1 = [1]$, $A_3 = [3]$, $A_5 = [5]$, respectively, by solving for their projective locations, as follows:

$$[1] = [u4v6] = [4][uv6] - [u][4v6]\,, \tag{64a}$$
$$[3] = [u2w6] = [w][u26] - [6][u2w]\,, \tag{64b}$$
$$[5] = [v2w4] = [2][vw4] - [v][2w4]\,. \tag{64c}$$

These can be rewritten as

$$[u][4v6] = [4][uv6] - [1]\,, \tag{65a}$$
$$[w][u26] = [6][u2w] + [3]\,, \tag{65b}$$
$$[v][2w4] = [2][vw4] - [5]\,. \tag{65c}$$

Now we take the scalar product[18] of the last three equations to get

$$\alpha[uvw] = ([4][uv6] - [1]) \cdot \{([2][vw4] - [5]) \times ([6][u2w] + [3])\}\,. \tag{66}$$

where $\alpha = [4v6][2w4][u26]$. Now, $\alpha \neq 0$ because none of its three factors is zero, according to Lemma 2. To expand the RHS[19] of this, we first distribute the cross products. In doing so, we lose the term containing [426] because of (62b), and the term containing [153] because of (62a).

Expanding, we get

$$\begin{aligned}[uvw] \doteq {} & [423][uv6][vw4] - [456][uv6][u2w] - [453][uv6] \\ & - [126][vw4][u2w] - [123][vw4] + [156][u2w]\,,\end{aligned} \tag{67}$$

where the overdot on the equal sign means that LHS of (67) is equal to the RHS up to the nonzero scalar factor α. Ordinarily, such a condition is a truism, but it works here since all we want to show is that is the RHS is zero, then so is also the LHS, so that the nonzero factor α is irrelevant.

STEP 3 TO THE PROOF: Conform the vector constraints to use in the scalar equation (67) and simplify in successive steps.

We observe that the terms in (67) are scalars, but the constraints we must invoke to finish this proof are vectors. Thus, to consistacize them, the simpler thing to do is to scalarize the vector constraints by either 1) crossing them with one vector and then dotting that result with another, or else 2) by dotting them with a cross product (that is taking their 'scalar product'). In either case we end up with a scalar in the form: $x \cdot y \times z$.

[17] There are many ways to go about doing this. The way I chose to do this might not be the best way, but it worked.

[18] In this paper, *scalar product* means, when applied to **three** vectors, a, b, c, is $[abc] = [a] \cdot ([b] \times [c])$.

[19] RHS means 'right-hand side.'

Now, I want to scalarize equations (65a)–(65c) in such a way so as to employ as many of the constraints found in (63a)–(63f) as I can, because it's by these constraints that the Pappus figure is properly defined.[20] Applying [2␣3] to (65a), we get

$$[2u3][4v6] = [243][uv6] - [213]\,, \tag{68}$$

and then applying (63b), we have[21]

$$[243][uv6] - [213] = 0\,. \tag{69a}$$

Similarly, applying [1␣6] to (65c) and then applying (63c), we get

$$[126][vw4] - [156] = 0\,. \tag{69b}$$

Lastly, applying [4␣5] to (65b) and then applying (63f), we get

$$[465][u2w] + [435] = 0\,. \tag{69c}$$

Returning to (67), we see that, on factoring $[u2w]$ out of the fourth and sixth terms on the RHS, the two terms cancel each other because of (69b), leaving us with

$$[uvw] \doteq [423][uv6][vw4] - [456][uv6][u2w] - [453][uv6] - [123][vw4]\,. \tag{70}$$

By a similar process we see that the first and fourth terms cancel because of (69a), leaving us with, after factoring and using (69c),

$$[uvw] \doteq -([456][u2w] + [453])[uv6] = 0\,. \tag{71}$$

Thus

$$[uvw] = 0\,, \tag{72}$$

as we needed to show. ■

[20] In other words, it would be very strange to be able prove a fundamental result concerning the Pappus figure without employing the information contained in the constraint equations that define that figure.

[21] Since [3u2]=0 and [2u3]=-[3u2] then [2u3]=0.

Now we've arrived at a discussion of the Hestenes-Ziegler paper itself.

12 Section 2 of the paper: Geometric Algebra

Again, it's not my purpose to reproduce all the basics of geometric algebra, but rather to fill-in some of the less obvious steps. Thus, we start at Eq. (2.6) [p.29, p.5][22]

Let's begin with a k-blade $M_k = a_1 \wedge \ldots \wedge a_k$, where the a_i's are vectors. The dagger operator reverses the ordering of the vectors in the wedge product, thus

$$M_k^\dagger = a_k \wedge \ldots \wedge a_1 \,. \tag{73}$$

So, what's the relation of $M_k^\dagger$ to M_k? To help us answer this, we begin with a lemma: the sum of integers from 1 to n is given as

$$\sum_{j=1}^{n} = 1 + 2 + \cdot + n = \frac{(n+1)n}{2} \,. \tag{74}$$

Hence,

$$\begin{aligned} M_k^\dagger &= a_k \wedge \ldots \wedge a_1 \\ &= (-1)^{k-1} a_1 \wedge a_k \wedge \ldots \wedge a_2 \\ &= (-1)^{(k-1)+(k-2)} a_1 \wedge a_2 \wedge a_k \wedge \ldots \wedge a_3 \\ &= \cdots \\ &= (-1)^{(k-1)+(k-2)+\cdots+1} a_1 \wedge a_2 \cdots \wedge a_k \\ &= (-1)^{k(k-1)/2} M_k \,. \end{aligned} \tag{75}$$

Next, we have Eq. (2.13) [p.30, p.6]

$$a \cdot B \equiv \langle\, aB \,\rangle_{s-1} = (-1)^{s-1} B \cdot a \tag{76}$$

where a is a vector and $B = \langle\, B \,\rangle_s$.

Before I start the proof, I want to state that the value of $(-1)^K$ for some integer K is highly restricted, meaning that its value depends only on whether

[22] I'm providing the equation numbers for both the published version and the preprint version, in that order.

K is even or odd. Hence, we can often reduce a complex expression for K down to an equivalent simpler expression.

$$\begin{aligned} a \cdot B &= (a \cdot B)^{\dagger\dagger} = \langle\, aB \,\rangle_{s-1}^{\dagger\dagger} \\ &= \langle\, B^{\dagger} a \,\rangle_{s-1}^{\dagger} = (-1)^{s(s-1)/2} \langle\, Ba \,\rangle_{s-1}^{\dagger} \\ &= (-1)^{s(s-1)/2} (-1)^{(s-1)(s-2)/2} \langle\, Ba \,\rangle_{s-1} \\ &= (-1)^{[s(s-1)+(s-1)(s-2]/2} B \cdot a \\ &= (-1)^{(s-1)^2} B \cdot a \\ &= (-1)^{s-1} B \cdot a \,. \end{aligned} \tag{77}$$

An extra credit problem: Show that

$$a \cdot (b \wedge B) = a \cdot b \, B - b \wedge (a \cdot B) \,, \tag{78}$$

where B is a k-blade.

$$\begin{aligned} a \cdot (b \wedge B) &= a \cdot (b \wedge b_1 \wedge b_2 \wedge \cdots b_k) \\ &= a \cdot b \, b_1 \wedge b_2 \wedge \cdots \wedge b_k - a \cdot b_1 \, b \wedge b_2 \wedge \cdots b_k \\ &\qquad\qquad + a \cdot b_2 b \wedge b_1 \wedge b_3 \wedge \cdots \wedge b_k - \cdots \\ &= a \cdot b \, B - a \cdot b_1 \, b \wedge b_2 \wedge \cdots \wedge b_k + a \cdot b_2 \, b \wedge b_1 \wedge b_3 \wedge \cdots \wedge b_k - \cdots \\ &= a \cdot b \, B - b \wedge [a \cdot b_1 \, b_2 \wedge \cdots \wedge b_k - a \cdot b_2 \, b_1 \wedge b_3 \wedge \cdots \wedge b_k + \cdots] \\ &= a \cdot b \, B - b \wedge (a \cdot B) \,. \end{aligned} \tag{79}$$

Next, we have Eq. (2.19) [p.31, p.6] .

$$\widetilde{A} \equiv AI^{-1} = A \cdot I^{-1} = (-1)^{r(n-r)} I^{-1} A \tag{80}$$

where $A = \langle\, A \,\rangle_r$ and $I = \langle\, I \,\rangle_n$.

$$\begin{aligned} A \cdot I^{-1} &= (A \cdot I^{-1})^{\dagger\dagger} = \langle\, AI^{-1} \,\rangle_{n-r}^{\dagger\dagger} = \langle\, I^{-1\dagger} A^{\dagger} \,\rangle_{n-r}^{\dagger} \\ &= (-1)^{n(n-1)/2} (-1)^{r(r-1)/2} \langle\, I^{-1} A \,\rangle_{n-r}^{\dagger} \\ &= (-1)^{n(n-1)/2} (-1)^{r(r-1)/2} (-1)^{(n-r)(n-r-1)/2} \langle\, I^{-1} A \,\rangle_{n-r} \\ &= (-1)^{n(n-1)+r^2-nr} I^{-1} A \\ &= (-1)^{r(r-n)} I^{-1} A \\ &= (-1)^{r(n-r)} I^{-1} A \,, \end{aligned} \tag{81}$$

where we used the fact that the expression $n(n-1)$ is always even and that the overall sign of the exponent to (-1) doesn't matter.

Next, we have Eq. (2.20) [p.31, p.7] and $B = \langle\, B \,\rangle_s$

$$A \cdot (BI) = (A \wedge B) I = (-1)^{s(n-s)} (AI) \cdot B \,. \tag{82}$$

We'll establish the first equality first.

$$
\begin{aligned}
A\cdot(BI) &= A\cdot(B\cdot I)\\
&= \langle\, A\cdot(B\cdot I)\,\rangle_{(n-s)-r}\\
&= \langle\, A(B\cdot I)\,\rangle_{(n-s)-r}\\
&= \langle\, A(BI)\,\rangle_{(n-s)-r}\\
&= \langle\, (AB)I\,\rangle_{n-(r+s)}\\
&= (A\wedge B)I\,. \qquad (83)
\end{aligned}
$$

Now for the second equality, using (2.19) to change the order of IB.

$$
\begin{aligned}
A\cdot(BI) &= (-1)^{s(n-s)}A\cdot(IB)\\
&= (-1)^{s(n-s)}\langle\, AIB\,\rangle_{n-r-s}\\
&= (-1)^{s(n-s)}\langle\, (AI)\cdot B\,\rangle_{n-r-s}\\
&= (-1)^{s(n-s)}(AI)\cdot B\,. \qquad (84)
\end{aligned}
$$

Next, we have Eq. (2.21) [p.31, p.7]

$$
(A\wedge B)^{\sim} = A\cdot\widetilde{B} = (-1)^{s(n-s)}\widetilde{A}\cdot B \qquad (85)
$$

So,

$$
(A\wedge B)^{\sim} = (A\wedge B)I^{-1} = (-1)^{s(n-s)}(AI)\cdot B\,. \qquad (86)
$$

And then,

$$
(A\wedge B)I^{-1} = A\cdot(BI^{-1}) = A\cdot\widetilde{B}\,. \qquad (87)
$$

Next, we have Eq. (2.25) [p.32, p.8]

Given that

$$
M\times N \equiv \tfrac{1}{2}(MN-NM)\,, \qquad (88)
$$

show that

$$
L\times(MN) = (L\times M)N + M(L\times N)\,. \qquad (89)
$$

Okay,

$$
\begin{aligned}
L\times(MN) &= \tfrac{1}{2}(LMN-MNL)\\
&= \tfrac{1}{2}(LMN-MLN+MLN-MNL)\\
&= \tfrac{1}{2}(LM-ML)N + M\tfrac{1}{2}(LN-NL)\\
&= (L\times M)N + M(L\times N)\,. \qquad (90)
\end{aligned}
$$

From *Clifford Algebra to Geometric Calculus* [3], p. 8, Equations (1.30a) and (1.30d), we have

$$
\begin{aligned}
a\cdot A_+ &= \tfrac{1}{2}(aA_+ - A_+a)\,,\\
a\wedge A_- &= \tfrac{1}{2}(aA_- - A_-a)\,, \qquad (91)
\end{aligned}
$$

where A_+ and A_+ are, respectively, the even and odd parts of the multivector A. Hence,

$$\begin{aligned} a \cdot A_+ &= a \times A_+ \,, \\ a \wedge A_- &= a \times A_- \,. \end{aligned} \tag{92}$$

The equation just after (2.29) [p.32, p.8] is

$$a \cdot (b \wedge B) = a \cdot b\, B + b \wedge (a \cdot B)\,, \tag{93}$$

but I got in (78)

$$a \cdot (b \wedge B) = a \cdot b\, B - b \wedge (a \cdot B)\,, \tag{94}$$

which is consistent with Eq. (1.42) p.12 of *Clifford Algebra to Geometric Calculus* (CAGC) [3], which is given as

$$a \cdot (A_r \wedge B) = (a \cdot A_r) \wedge B + (-1)^r A_r \wedge (a \cdot B)\,, \tag{95}$$

where $B = \langle B \rangle_k$ and $k \geq r \geq 1$.

If we now swap a with b in (78), we get

$$b \cdot (a \wedge B) = b \cdot a\, B - a \wedge (b \cdot B)\,. \tag{96}$$

Now, if we subtract (96) from (78), we get

$$a \cdot (b \wedge B) - b \cdot (a \wedge B) = a \wedge (b \cdot B) - b \wedge (a \cdot B)\,. \tag{97}$$

However, this has a sign difference compared to Eq. (2.29) in the paper [p.32, p.8].

Next, we have (2.30) [p.32, p.8] is

$$AB = A \cdot B + A \times B + A \wedge B\,, \tag{98}$$

where A is a bivector and B is an s-vector. The book *CAGC* [3] gives the following proof from page 10 (1.37a), where $A_2 = a_1 a_2 = a_1 \wedge a_2$:

$$\begin{aligned} A_2 B_s &= a_1 a_2 B_s = a_1 (a_2 \cdot B_2 + a_2 \wedge B_s) \\ &= a_1 \cdot (a_2 \cdot B_s) + a_1 \wedge (a_2 \cdot B_s) + a_1 \cdot (a_2 \wedge B_s) + a_1 \wedge (a_2 \wedge B_s) \\ &= A_2 \cdot B_s + [\, a_1 \wedge (a_2 \cdot B_s) + a_1 \cdot (a_2 \wedge B_s)\,] + A_2 \wedge B_s \\ &= A_2 \cdot B_s + \langle A_2 B_s \rangle_s + A_2 \wedge B_s\,. \end{aligned} \tag{99}$$

If we now switch the order of A_2 and B_s we get

$$B_s A_2 = B_s \cdot A_2 + \langle B_s A_2 \rangle_s + B_s \wedge A_2\,. \tag{100}$$

On subtracting this last equation from the previous one, we get

$$\begin{aligned} A_2 B_s - B_s A_2 = A_2 \cdot B_s - B_s \cdot A_2 &+ \langle A_2 B_s - B_s A_2 \rangle_s \\ &+ A_2 \wedge B_s - B_s \wedge A_2\,. \end{aligned} \tag{101}$$

That the last two terms cancel is easy to show. Just move a_2 through B_s and then move a_2 through it, and find that $A_2 \wedge B_s = B_s \wedge A_2$. That leaves us with

$$A_2 B_s - B_s A_2 = A_2 \cdot B_s - B_s \cdot A_2 + \langle\, A_2 B_s - B_s A_2 \,\rangle_s \,. \tag{102}$$

Next, we'll show that

$$A_2 \cdot B_s = B_s \cdot A_2 \,. \tag{103}$$

Thus,

$$\begin{aligned} A_2 \cdot B_s &= (a_1 \wedge a_2) \cdot B_s \\ &= a_1 \cdot (a_2 \cdot B_s) \\ &= a_1 \cdot [\,(-1)^{s-1} B_s \cdot a_2\,] \\ &= (-1)^{s-1} a_1 \cdot \langle\, B_s \cdot a_2 \,\rangle_{s-1} \\ &= (-1)^{s-1} (-1)^{s-2} \langle\, B_s \cdot a_2 \,\rangle_{s-1} \cdot a_1 \\ &= (-1)^{2s-3} B_s \cdot (a_2 \wedge a_1) \\ &= (-1)^{2s-2} B_s \cdot (a_1 \wedge a_2) \\ &= B_s \cdot A_2 \,. \end{aligned} \tag{104}$$

Therefore, (102) becomes

$$A_2 B_s - B_s A_2 = \langle\, A_2 B_s - B_s A_2 \,\rangle_s \,. \tag{105}$$

That leaves us to show that

$$\langle\, A_2 B_s \,\rangle_s = -\langle\, B_s A_2 \,\rangle_s \,. \tag{106}$$

Hence,

$$\begin{aligned} \langle\, A_2 B_s \,\rangle_s &= a_1 \wedge (a_2 \cdot B_s) + a_1 \cdot (a_2 \wedge B_s) \\ &= (-1)^{s-1} a_1 \wedge (B_s \cdot a_2) + (-1)^{(s+1)-1} (a_2 \wedge B_s) \cdot a_1 \\ &= (-1)^{s-1} (-1)^{s-1} (B_s \cdot a_2) \wedge a_1 + (-1)^s (-1)^s (B_s \wedge a_2) \cdot a_1 \\ &= (B_s \cdot a_2) \wedge a_1 + (B_s \wedge a_2) \cdot a_1 \\ &= \langle\, B_s A_2^\dagger \,\rangle_s \\ &= -\langle\, B_s A_2 \,\rangle_s \,. \end{aligned} \tag{107}$$

Therefore,

$$A \times B \equiv \tfrac{1}{2}(AB - BA) = \tfrac{1}{2}(\langle\, A_2 B_s \,\rangle_s - \langle\, B_s A_2 \,\rangle_s) = \langle\, A_2 B_s \,\rangle_s \,, \tag{108}$$

which finalizes our proof of (98).

Next, we have (2.31) [p.33, p.8]. For 2-blades A, B, C:

$$(A \times B) \cdot C = \langle\, ABC \,\rangle = \langle\, CAB \,\rangle = (C \times A) \cdot B \,. \tag{109}$$

Lemma

For multivectors M, N

$$\langle MN \rangle = \langle NM \rangle . \tag{110}$$

The proof of this is based on two simple facts. First, for every scalar S, $S = S^\dagger$. Hence,

$$\langle MN \rangle = \langle MN \rangle^\dagger = \langle N^\dagger M^\dagger \rangle . \tag{111}$$

The second fact is that

$$\langle MN \rangle = \langle \sum_k M_k \cdot N_k \rangle , \tag{112}$$

where k sums over all grade values in the multivectors, and only the scalar values are retained. Therefore,

$$\begin{aligned} \langle MN \rangle = \langle N^\dagger M^\dagger \rangle &= \langle \sum_k N_k^\dagger \cdot M_k^\dagger \rangle \\ &= (-1)^{k(k-1)/2}(-1)^{k(k-1)/2} \langle \sum_k N_k \cdot M_k \rangle \\ &= (-1)^{k(k-1)} \langle \sum_k N_k \cdot M_k \rangle = \langle NM \rangle . \end{aligned} \tag{113}$$

Now,

$$\begin{aligned} \langle ABC \rangle &= \langle (A \cdot B + A \times B + A \wedge B) C \rangle \\ &= \langle (A \times B) C \rangle \\ &= \langle \langle A \times B \rangle_2 C \rangle \\ &= (A \times B) \cdot C . \end{aligned} \tag{114}$$

For what it's worth, I have a second proof.

$$\begin{aligned} \langle ABC \rangle &= \tfrac{1}{2} \langle ABC \rangle + \tfrac{1}{2} \langle ABC \rangle^\dagger \\ &= \tfrac{1}{2} \langle ABC \rangle + \tfrac{1}{2} \langle C^\dagger B^\dagger A^\dagger \rangle \\ &= \tfrac{1}{2} \langle ABC \rangle - \tfrac{1}{2} \langle CBA \rangle \\ &= \tfrac{1}{2} \langle ABC \rangle - \tfrac{1}{2} \langle BAC \rangle \\ &= \langle \tfrac{1}{2} (AB - BA) C \rangle \\ &= \langle (A \times B) C \rangle \\ &= (A \times B) \cdot C . \end{aligned} \tag{115}$$

We're almost finished now. On cyclically permuting the variables in the last equation, we get

$$\langle CAB \rangle = (C \times A) \cdot B . \tag{116}$$

And since $\langle\, ABC\,\rangle = \langle\, CAB\,\rangle$ then,

$$(A \times B) \cdot C = (C \times A) \cdot B\,. \tag{117}$$

From (97), we get

$$(a \wedge b) \times B = a \cdot (b \wedge B) - b \cdot (a \wedge B)\,. \tag{118}$$

Hence,

$$C \times B = (c \wedge c') \times B = c \cdot (c' \wedge B) - c' \cdot (c \wedge B)\,. \tag{119}$$

$$\langle\, ABC\,\rangle = \langle\, A(B \times C)\,\rangle = -\langle\, A(C \times B)\,\rangle\,. \tag{120}$$

So,

$$\begin{aligned}
\langle\, ABC\,\rangle &= -\langle\, A(C \times B)\,\rangle \\
&= -\langle\, A(c \cdot (c' \wedge B) - c' \cdot (c \wedge B))\,\rangle \\
&= \langle\, A[\,c' \cdot (c \wedge B) - c \cdot (c' \wedge B)\,]\,\rangle \\
&= (A \wedge c') \cdot (c \wedge B) - (A \wedge c) \cdot (c' \wedge B) \\
&= (A \wedge c') \cdot (B \wedge c) - (A \wedge c) \cdot (B \wedge c')\,.
\end{aligned} \tag{121}$$

[I still have a sign difference here.]

13 Section 3 of the paper: The Algebra of Incidence

Now we're at [p.33, p.10].

It's time now to meet the *meet*. The meet of two blades A and B is represented as $A \vee B$. We can ask if this new operation can be expressed in terms of operations we already know. The answer is yes. We need only the join operator and duality.

$$(A \vee B)^{\sim} \equiv \widetilde{A} \wedge \widetilde{B}\,, \tag{122}$$

which is Eq. (3.5). From the definition of the dual:

$$(A \vee B)^{\sim} = (A \vee B)I^{-1} = \widetilde{A} \wedge \widetilde{B}\,. \tag{123}$$

From this we get the first part of (3.6).

$$A \vee B = (\widetilde{A} \wedge \widetilde{B})I\,. \tag{124}$$

Now, let $\widetilde{A} = \langle\, \widetilde{A}\,\rangle_p$ and $\widetilde{B} = \langle\, \widetilde{B}\,\rangle_q$. Then,

$$\begin{aligned}
(\widetilde{A} \vee \widetilde{B})I &= \langle\, (AI^{-1}) \wedge (BI^{-1})\,\rangle_{p+q} I \\
&= \langle\, (AI^{-1})(BI^{-1})I\,\rangle_{n-(p+q)} \\
&= \langle\, (AI^{-1})B\,\rangle_{(n-q)-p} \\
&= \langle\, \widetilde{A}B\,\rangle_{(n-q)-p} \\
&= \widetilde{A} \cdot B\,,
\end{aligned} \tag{125}$$

where in the next-to-last line, the grade $(n-q)-p$ had p associated to $\widetilde{A}$ and $n-q$ associated to B, and whose difference would give us an inner product.

Now we assign: $A = \langle A \rangle_r$ and $B = \langle B \rangle_s$. Next, we employ Eq. (2.12) of the paper, using appropriate swapping of variables:

$$\begin{aligned}
\widetilde{A} \cdot B &= \langle (AI^{-1})B \rangle_{(n-r)-s} \\
&= \langle A[I^{-1} \cdot B] \rangle_{(n-s)-r} \\
&= (-1)^{s(n-s)} \langle A[B \cdot I^{-1}] \rangle_{(n-s)-r} \\
&= (-1)^{s(n-s)} \langle A[BI^{-1}] \rangle_{(n-s)-r} \\
&= (-1)^{s(n-s)} A \cdot \widetilde{B}\,.
\end{aligned} \tag{126}$$

Next we have (3.7), [p.34, p.10]. We begin with

$$(A \vee D)^{\sim} = (\widetilde{A} \wedge \widetilde{D})\,. \tag{127}$$

So, we set $D = B \vee C$, and get

$$\begin{aligned}
(A \vee (B \vee C))^{\sim} &= (\widetilde{A} \wedge (B \vee C)^{\sim}) \\
&= (\widetilde{A} \wedge (\widetilde{B} \wedge \widetilde{C})) \\
&= (\widetilde{A} \wedge \widetilde{B} \wedge \widetilde{C}) \\
&= ((\widetilde{A} \wedge \widetilde{B}) \wedge \widetilde{C}) \\
&= ((A \vee B) \vee C)^{\sim}\,.
\end{aligned} \tag{128}$$

Thus, we have shown that the meet operator is associative, which followed from the fact that the wedge operator is associative.

In the paragraph just prior to (3.8) in the preprint paper, the equation $A \wedge B = A \times B$ should be $\widetilde{A} \wedge \widetilde{B} = \widetilde{A} \times \widetilde{B}$.

Next, we have (3.8) [p.35, p.10]. For $(n-1)$-blades A, B:

$$A \vee B = (-1)^{n-1}(A \times B)I^{-1}\,. \tag{129}$$

Hence,

Note: $r = s = n-1$. Then, from (2.12, with $s(n-s) = n-1$:

$$B \cdot I = (-1)^{n-1} I \cdot B\,, \tag{130}$$

and

$$A \cdot I = (-1)^{n-1} I \cdot A\,. \tag{131}$$

Also, in any expression that has a factor of both I and I^{-1}, they can be exchanged. For example,

$$(d \wedge e\,I)\,xy\,I^{-1} = (d \wedge e\,I^{-1})\,xyI\,. \tag{132}$$

Why is this? Because I and I^{-1} differ only by a scalar multiple, and which factor has this scalar multiple doesn't matter.

$$\begin{aligned}
A \vee B &= (\widetilde{A} \wedge \widetilde{B})I = [\,(AI^{-1}) \wedge (BI^{-1})\,]I \\
&= \tfrac{1}{2}\langle\,(AI^{-1})(BI^{-1}) - (BI^{-1})(AI^{-1})\,\rangle_2 I \\
&= \tfrac{1}{2}\langle\,(AI^{-1})(BI) - (BI^{-1})(AI)\,\rangle_2 I^{-1} \\
&= (-1)^{n-1}\tfrac{1}{2}\langle\,(AI^{-1})(I \cdot B) - (BI^{-1})(I \cdot A)\,\rangle_2 I^{-1} \\
&= (-1)^{n-1}\tfrac{1}{2}\langle\,(AI^{-1})(IB) - (BI^{-1})(IA)\,\rangle_2 I^{-1} \\
&= (-1)^{n-1}\tfrac{1}{2}[\,(AB) - (BA)\,]I^{-1} \\
&= (-1)^{n-1}(A \times B)I^{-1}\,. \qquad (133)
\end{aligned}$$

At this point we have to take into consideration an important special case. Equations (3.5) and (3.6) are suitable to define the meet only for the case when $r + s > n$. If $r + s = n$, then $\widetilde{A} \cdot B = \langle\,\widetilde{A} \cdot B\,\rangle$, which is a scalar. But what are the allowable scalar values? The meet has to be a sub-vector space of the vector space of dimension n. However, the only scalar that could impersonate a zero-dimensional sub-vector space is zero. Hence, we have that

$$A \vee B = \widetilde{A} \cdot B \equiv 0 \quad \text{for } r + s = n\,. \qquad (134)$$

Let's look at an important special case. Say that we are in the projective plane and we know that three lines A, B, and C are concurrent at a point, say point D. Then what is $A \vee B \vee C$? Well, we already know that

$$A \vee B \vee C = (A \vee B) \vee C = (\widetilde{A} \cdot B) \vee C\,. \qquad (135)$$

Algebraically speaking, the lines A, B, and C are 2-blades, and $n = 3$. Thus, $r = 2$, $s = 2$, $t = 2$ for the steps of the blades A, B, and C, respectively. Therefore, $\widetilde{A}$ has grade 1, and so $D \equiv \widetilde{A} \cdot B$ also has grade 1. Hence, in the resulting expression $D \vee C$ the sum of their grades is 3. Therefore, we have to assign $D \vee C = 0$, which forces us to conclude that

$$A \vee B \vee C = 0\,. \qquad (136)$$

Next we move to the projective interpretation of blades and incidence relations, beginning at [p.35, p.10]. I'll start my comments at Eq. (3.10) and the paragraph just before it. The text claims

> For example, each pair of distinct points a and b determine unique line $a \wedge b$. Also, a point p lies on a line A if and only if
>
> $$p \wedge A = 0\,.$$
>
> [This is Eq. (3.10) in the paper.]

So, I'll now explain this as I understand it, by using a projective plane in two dimensions $\mathcal{P}_2$. We'll be considering figures in a plane $\mathcal{P}_2$ which by construction contain points and lines. I already explained how to interpret this in the Gibbs's vector-algebra representation given before. My explanation here will keep close to that previous explanation.

We begin with a 3-D vector space. By definition this vector space has a special point, called the zero vector, $\mathbf{0}$. All the vectors of this space have the base point of these vectors at this zero point. Every point in this underlying space has a vector whose tip is at this point. Thus, there is a one-to-one correspondence between the points in the underlying space and the vectors of the vector space.

Next, we embed a plane into this space, which we'll call the projective plane $\mathcal{P}_2$. It can be any plane you wish but it cannot contain the zero point. Now, imagine that there are two distinct points on $\mathcal{P}_2$, call them a and b. The join of a and b will be a unique line in $\mathcal{P}_2$ containing these points.

Here's where the magic happens. That line that contains a and b can be represented by the intersection of $\mathcal{P}_2$ with some plane that contains the origin $\mathbf{0}$. Such a plane I refer to as an *oriplane*. It just so happens that the bivector $a \wedge b$ has the same direction in space as does the oriplane that contains points a and b and $\mathbf{0}$. Hence, we can represent this oriplane by the bivector $a \wedge b$. The only thing special about vectors a and b is that they 'are distinct vectors that lie in the oriplane. Any other two distinct nonzero vectors that 'lie in the same oriplane' would also serve to represent both the oriplane and its intersection with the line in $\mathcal{P}_2$ which contain the points a and b. Thus $a \wedge b$ represents at the same time an oriplane and the join of the points $a \wedge b$.

Now, the set of all points x on this join line must satisfy the relation

$$x \wedge a \wedge b = 0\,,$$

for to do so is to mean that the three vectors a, b, x as vectors in 3-space are coplanar. Hence, for a particular point p in $\mathcal{P}_2$, it lies on the join of a and b, with $A \equiv a \wedge b$, if and only if

$$p \wedge A = 0\,.$$

Okay, so I used the bivector A as constructed from vectors a and b, but any nonzero bivector of the same plane would do. And those two bivectors would be nonzero scalar multiples of each other.

Now, we move to Eq. (3.10) and the 'absorption relation'

$$p \vee A = p\,, \tag{137}$$

where 'p lies on line A' is the projective plane interpretation. As viewed from outside the projective plane, we interpret it as the vector p lying within the bivector A (or rather the oriplane it represents). If we interpret (137) as the projective line $\mathcal{P}_1$ and use (3.6), we get that

$$p \vee A = \widetilde{p} \cdot A = \widetilde{p} A = p\,. \tag{138}$$

In other words, if $\mathcal{P}_1$ is the space of interest, then its pseudoscalar is the bivector A. Therefore

$$\widetilde{p} = pA^{-1} = p \cdot A^{-1}\,, \tag{139}$$

whereby the bivector A^{-1} will rotate the vector p by 90° clockwise in the A plane. Therefore

$$\widetilde{p} \cdot p = 0\,. \tag{140}$$

Returning to (138), we have

$$p \vee A = \widetilde{p} \cdot A = \langle\, pA^{-1} \cdot A\,\rangle_1 = \langle\, pA^{-1}A\,\rangle_1 = \langle\, p\,\rangle_1 = p\,. \tag{141}$$

Put into words, multiplying p by A^{-1} will rotate the vector p by 90° clockwise with a scalar factor. And then multiplying pA^{-1} by A on the right will rotate that vector by 90° counterclockwise and undo the scalar factor.

Now we're at [p.36, p.12] for incidence relations among lines

We begin with Eq. (3.12), with two distinct lines being represented by A and B. Then, these lines intersect if and only if

$$A \wedge B = 0\,. \tag{142}$$

As I explained in the Gibbs's vector representation, we can look at the lines in $\mathcal{P}_2$ the intersections (meets) of oriplanes with the plane $\mathcal{P}_2$. For the case of parallel distinct lines in $\mathcal{P}_2$, their oriplanes will intersect in a line that is parallel to $\mathcal{P}_2$ and goes through the origin. For two distinct lines in $\mathcal{P}_2$ that are not parallel, they will intersect in a point, as in the Euclidean case, but we have the advantage of projective geometry to find that point of intersection. With the Gibbs's vectors, we know from our earlier mathematical studies that two non-parallel planes in 3-D will intersect in a line whose vector representation is given as the cross product of the two normals of the two planes.[23] However, in the geometric algebra, instead of using cross products, we'll use the bivectors to the planes to represent the planes.

So, my analysis will be exclusively in $\mathcal{P}_2$. Thus, the two 'lines' A and B meet at point

$$p = A \vee B = (A \times B)J^{-1}\,, \tag{143}$$

where we have used (129) and J is the join of A and B, which is the smallest proper space that contains both A and B. If we assign particular factored bivectors to both A and B as

$$A = a \wedge a' \quad \text{and} \quad B = b \wedge b'\,, \tag{144}$$

then we get Eq. (3.14)

$$A \vee B = \widetilde{A} \cdot B = (\widetilde{A} \cdot b)b' - (\widetilde{A} \cdot b')b = [aa'b]b' - [aa'b']b\,, \tag{145}$$

[23] In the Gibbs's part of the analysis, I referred to this point as the 'projective location'.

where we used (2.22). This computation is the same that I got in the Gibbs's version that I did above, such as in Eq. (54a), except that I got an overall negative sign in my version compared to the above. This may be due to the difference of how the cross product is derived from a bivector. I'll have to check into this later on.

Let's go over this computation. But first, we take a look at this bracket notation $[xyz]$ is what I have call the 'scalar product of the three vectors' x, y, z, or,

$$[xyz] = [(x \times y) \cdot z] = [(z \times x) \cdot y] = [(y \times z) \cdot x]\,. \tag{146}$$

Hence,

$$[xyz] = [zxy] = [yzx]\,. \tag{147}$$

Also, we have that

$$\widetilde{A} = AJ^{-1} = (a \wedge a')J^{-1} = a \times a'\,. \tag{148}$$

Therefore,

$$\begin{aligned} A \vee B &= \widetilde{A} \cdot B = \widetilde{A} \cdot (b \wedge b') \\ &= (\widetilde{A} \cdot b)b' - (\widetilde{A} \cdot b')b \\ &= ((a \times a') \cdot b)b' - ((a \times a') \cdot b')b \\ &= [aa'b]b' - [aa'b']b\,. \end{aligned} \tag{149}$$

We can express the meet in terms of the projective location:

$$\begin{aligned} A \vee B &= [aa'b]b' - [aa'b']b \\ &= b'[aa'b] - b[aa'b'] \\ &= b'[baa'] - b[b'aa'] \\ &= [bb'aa'] = -[aa'bb']\,. \end{aligned} \tag{150}$$

In the problem I did above on Pappus, I defined the meet of two lines $a \wedge a'$ and $b \wedge b'$ into a point as the 'projective location' of two lines in $\mathcal{P}_2$ by $[aa'bb'] = (a \times a') \times (b \times b')$. This differs by a minus sign from how the meet is defined here. But that should be fine because overall sign is nothing more than a nonzero scalar multiple, which we've been given to ignore. Indeed, if one demands that

$$A \vee B = B \vee A\,, \tag{151}$$

say in $\mathcal{P}_2$, then this sign difference is obvious, for

$$\begin{aligned} B \vee A &= \widetilde{B} \cdot A = (BI^{-1}) \cdot A = \langle\, BI^{-1}A \,\rangle_1 \\ &= \langle\, BAI^{-1} \,\rangle_1 \\ &= \langle\, B \times AI^{-1} \,\rangle_1 = -\langle\, A \times BI^{-1} \,\rangle_1 \\ &= -\langle\, ABI^{-1} \,\rangle_1 = -\langle\, AI^{-1}B \,\rangle_1 \\ &= -\widetilde{A} \cdot B \\ &= -A \vee B\,. \end{aligned} \tag{152}$$

We have seen that one way to declare three lines in $\mathcal{P}_2$ to be concurrent (at a point) is to use condition (136). Seemingly a more practical method is to use Eq. (3.15). Let p be the meet of two lines A and B. Then p is on a third line C if

$$p \wedge C = 0\,. \tag{153}$$

Viewed from the enveloping space, this means that the vector p lies in the plane specified by the bivector C. So, let $p = A \vee B$, then

$$p \wedge C = (A \vee B) \wedge C = 0\,. \tag{154}$$

On expanding this, we get

$$\begin{aligned}(A \vee B) \wedge C &= [\,(A \times B)J^{-1}\,] \wedge C \\ &= [\,(A \times B) \cdot C\,]J^{-1}\,.\end{aligned} \tag{155}$$

So, if A, B, and C are all bivectors then

$$(A \times B) \cdot C = \langle\, ABC \,\rangle = 0\,. \tag{156}$$

is the simple algebraic condition that lines A, B, and C are concurrent.

So, we might ask what $\langle\, ABC \,\rangle$ should really mean. Should it mean $\langle\, ABC \,\rangle = A \cdot (B \times C)$ for example?

$$\begin{aligned}A \cdot (B \times C) &= \tfrac{1}{2}\langle\, A(BC - CB) \,\rangle = \tfrac{1}{2}[\,\langle\, ABC \,\rangle - \langle\, ACB \,\rangle\,] \\ &= \tfrac{1}{2}[\,\langle\, ABC \,\rangle - \langle\, BAC \,\rangle\,] \\ &= \tfrac{1}{2}\langle\, ABC - BAC \,\rangle \\ &= \tfrac{1}{2}\langle\, (AB - BA)C \,\rangle \\ &= (A \times B) \cdot C\,.\end{aligned} \tag{157}$$

14 Section 4 of the paper: Two-dimensional projective geometry

Now we're at [p.37, p.13].

So, at this point we confine our interest just to $\mathcal{P}_2$. To begin with, we already saw that three points in $\mathcal{P}_2$, say a, b, and c, are on a line if and only if their vector representations lie in some oriplane, in which case

$$J = a \wedge b \wedge c = 0\,. \tag{158}$$

On the other hand, if a, b, and c are three distinct non-collinear points in $\mathcal{P}_2$ then $J = a \wedge b \wedge c$ will not be zero, and will, in fact, be a scalar multiple of the pseudoscalar in G_3.

Of some usefulness to us is the fact that every bivector in G_3 is a 2-blade. Let's prove this. It's sufficient to prove this if the dual of every vector is a

2-blade, which is obvious. Now, we assume that the there exists a bivector B that cannot be written as the dual of a vector. We'll make the simplifying assumption that

$$B = B_1 + B_2 \,. \tag{159}$$

So, what is the dual of B? Let the pseudoscalar of G_3 be represented by I. Then the dual of B is BI, yielding

$$BI = B_1 I + B_2 I \,. \tag{160}$$

But the dual of every bivector is some vector, hence $B_1 I = b_1$ and $B_2 I = b_2$, then

$$BI = B_1 I + B_2 I = b_1 + b_2 = b \,, \tag{161}$$

where b is just some vector. Hence B is the dual of a vector, and thus is a 2-blade.

Now, three distinct non-collinear points in G_3, a, b, and c, which are in $\mathcal{P}_2$, generate in pairs three distinct lines in $\mathcal{P}_2$, which correspond to three distinct oriplanes, which corresponds to three distinct bivectors in G_3. Now comes the big step: we will calculate with objects in G_3, though we will refer to their corresponding objects in $\mathcal{P}_2$. Thus, we have the three 'lines'

$$A = b \wedge c = \widetilde{A} J \,, \quad B = c \wedge a = \widetilde{B} J \,, \quad C = a \wedge b = \widetilde{C} J \,. \tag{162}$$

Having the three points a, b, and c, we can define a pseudoscalar with them, such as

$$J = a \wedge b \wedge c \,. \tag{163}$$

Probably J is not a unit pseudoscalar, but that's okay. Anyway, we can now solve for the vectors dual to A, B, C, from (162) to get Eq. (4.3):

$$\widetilde{A} = AJ^{-1} = \frac{b \wedge c}{a \wedge b \wedge c} \,, \quad \widetilde{B} = BJ^{-1} = \frac{c \wedge a}{a \wedge b \wedge c} \,, \quad \widetilde{C} = CJ^{-1} = \frac{a \wedge b}{a \wedge b \wedge c} \,. \tag{164}$$

The division by J is legitimate because in G_3, J is a pseudoscalar and pseudoscalars commute with all elements of the G_3 algebra.

Now for Eq. (4.4). Since the vector a is a common factor of distinct bivectors B and C, then, with the help of (124):

$$a = C \vee B = (\widetilde{C} \wedge \widetilde{B}) J \,, \tag{165}$$

and similarly for vectors b and c. For Eq. (4.5a), I'll give a sample calculation:

$$\begin{aligned} a \cdot \widetilde{B} &= \langle\, a\widetilde{B} \,\rangle = \langle\, aBJ^{-1} \,\rangle \\ &= \langle\, aB \,\rangle_3 J^{-1} \\ &= \langle\, ac \wedge a \,\rangle_3 J^{-1} \\ &= 0 \,. \end{aligned} \tag{166}$$

For Eq. (4.5b), I'll give another sample calculation:

$$\begin{aligned} a \cdot \widetilde{A} &= \langle\, a\widetilde{A}\,\rangle = \langle\, aAJ^{-1}\,\rangle \\ &= \langle\, ab \wedge c\,\rangle_3 J^{-1} \\ &= \langle\, a \wedge b \wedge c\,\rangle_3 J^{-1} \\ &= JJ^{-1} \\ &= 1\,. \end{aligned} \tag{167}$$

For Eq. (4.6), the quick path to the solution is to use Eq. (2.32):

$$\begin{aligned} \langle\, CBA\,\rangle &= \langle\, CB(b \wedge c)\,\rangle \\ &= (C \wedge b) \cdot (B \wedge c) - (C \wedge c) \cdot (B \wedge b) \\ &= (a \wedge b \wedge b) \cdot (c \wedge a \wedge c) - (a \wedge b \wedge c) \cdot (c \wedge a \wedge b) \\ &= -(a \wedge b \wedge c) \cdot (a \wedge b \wedge c) \\ &= -J^2\,. \end{aligned} \tag{168}$$

The paper's result is J^2.

Next, Eq. (4.7).

$$\widetilde{C} \wedge \widetilde{B} \wedge \widetilde{A} = J^{-1}\,. \tag{169}$$

It's sufficient to prove (169) to prove that

$$(\widetilde{C} \wedge \widetilde{B} \wedge \widetilde{A})J = 1\,. \tag{170}$$

This equation becomes

$$\langle\, (\widetilde{C} \wedge \widetilde{B} \wedge \widetilde{A})J\,\rangle = 1\,. \tag{171}$$

The advantage of using this latter form is that we can add in or subtract out any quantity that will get removed when the scalar select operator does its duty. In other words, it will remove any quantity that's not a scalar.

$$\begin{aligned} \langle\, (\widetilde{C} \wedge \widetilde{B} \wedge \widetilde{A})J\,\rangle &= \langle\, (CJ^{-1}) \wedge (BJ^{-1}) \wedge (AJ^{-1})J\,\rangle \\ &= \langle\, (CJ^{-1}) \wedge (BJ^{-1})(AJ^{-1})J\,\rangle \\ &= \langle\, (CJ^{-1}) \wedge (BJ^{-1})A\,\rangle \\ &= \langle\, (CJ^{-1})(BJ^{-1})A\,\rangle \\ &= \langle\, CBAJ^{-2}\,\rangle \\ &= \langle\, CBA\,\rangle J^{-2} \\ &= J^2 J^{-2} \\ &= 1\,, \end{aligned} \tag{172}$$

where I used that $\langle\, CBA\,\rangle = J^2$.

So now we finally arrive at Desargues's Theorem! [p.39, p.14]

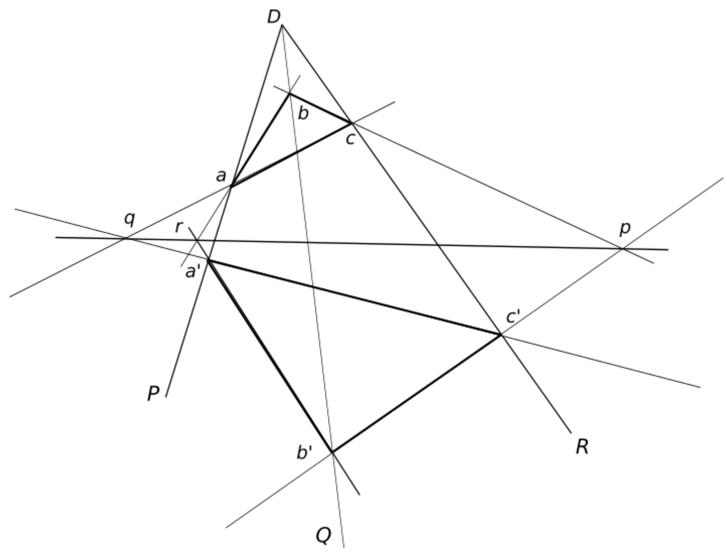

Figure 8. Desargues's Theorem: Two triangles that are perspective from a point are perspective from a line. As before, the figure lies in $\mathcal{P}_2$, which lies in a 3-dim vector space (which is hidden). And $\mathcal{P}_2$ does not contain the origin of this vector space. All the labeled points are the tips of vectors whose bases are at the origin.

With regards to Desargues's figure in Figure 8, we can add to relations (162), (163), and (164) the following:

$$A' = b' \wedge c' = \widetilde{A}' J' , \quad B' = c' \wedge a' = \widetilde{B}' J' , \quad C' = a' \wedge b' = \widetilde{C}' J' . \tag{173}$$

$$J' = a' \wedge b' \wedge c' . \tag{174}$$

Probably J is not a unit pseudoscalar, but that's okay. Anyway, we can now solve for the vectors dual to A', B', C', from (162) to get the results that correspond Eq. (4.3):

$$\widetilde{A}' = A' J'^{-1} = \frac{b' \wedge c'}{a' \wedge b' \wedge c'} , \; \widetilde{B}' = B' J'^{-1} = \frac{c' \wedge a'}{a' \wedge b' \wedge c'} , \; \widetilde{C}' = C' J'^{-1} = \frac{a' \wedge b'}{a' \wedge b' \wedge c'} . \tag{175}$$

As pseudoscalars for G_3, both J and J' have to represent $\mathcal{P}_2$. This seems reasonable enough since every pseudoscalar of G_3 has to represent every projective plane, since the projective plane is chosen arbitrarily, except that it not contain the origin. Thus,

$$J = a \wedge b \wedge c = [\, abc\,] I , \qquad J' = a' \wedge b' \wedge c' = [\, a'b'c'\,] I , \tag{176}$$

where I is the unit pseudoscalar of G_3, and

$$[\, abc\,] = a \cdot (b \times c) \quad \text{and} \quad [\, a'b'c'\,] = a' \cdot (b' \times c') . \tag{177}$$

Now, we need to define three lines P, Q, and R based on the figure (defined as the joins of pairs of points in $\mathcal{P}_2$):

$$P = a \wedge a' \,, \quad Q = b \wedge b' \,, \quad R = c \wedge c' \,, \tag{178}$$

which is Eq. (4.11).

What we're trying to show is that if lines are concurrent at some point, which we'll call D, then points p, r, and q are incident with the same line, which we'll call $\overline{(prq)}$, which is the join of points p, r, and q, if our theorem is correct. Thus,

$$p = A \vee A' \,, \quad q = B \vee B' \,, \quad r = C \vee C' \,. \tag{179}$$

At this point, we are looking for a relationship between the concurrence of the three lines P, Q, and R at point D and the joins of the three points p, r, and q. If lines P, Q, and R meet at a common point D, then we could naively conclude that

$$P \vee Q \vee R = D \,. \tag{180}$$

But we've already concluded that this double meet must be set to zero.

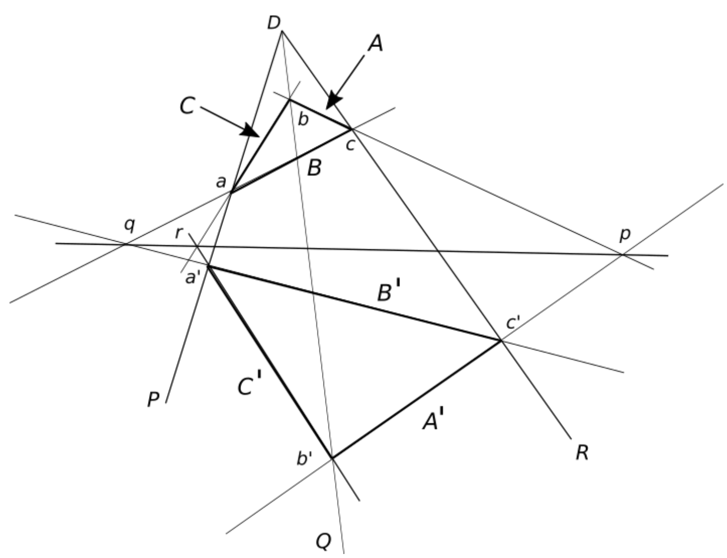

Figure 9. Desargues's Theorem: This version of the figure includes the six lines (joins) A, B, C, and A', B', C'.

But whether $P \vee Q \vee R$ is zero or not,

$$P \vee Q \vee R = (\widetilde{P} \wedge \widetilde{Q} \wedge \widetilde{R})\, I \,. \tag{181}$$

The LHS of this equation is (corresponds to the next displayed equation after Eq. (4.14) [p.40, p.15])

$$\langle\, (a \wedge a')(b \wedge b')(c \wedge c')\,\rangle = (b' \wedge b \wedge c) \cdot (a \wedge c' \wedge a') - (b \wedge b' \wedge c') \cdot (c \wedge a \wedge a') \,. \tag{182}$$

Now, we want to expand the RHS of (182), without grouping any vector with its prime, so, for example,

$$b' \wedge b \wedge c = b' \wedge (b \wedge c) = b' \wedge A = b' \wedge (\widetilde{A}J) \tag{183}$$

From Fig. 9, we see that $b' = A' \vee C' = \widetilde{A}' \wedge \widetilde{C}' J'$; therefore,

$$b' \wedge b \wedge c = b' \wedge (b \wedge c) = (\widetilde{A}' \wedge \widetilde{C}' J') \wedge (\widetilde{A}J) = (\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})JJ' \tag{184}$$

Note: Since J and J' are both pseudoscalars, their product $JJ' = J'J$ is a nonzero scalar. Let's fill-in some steps:

$$\begin{aligned} b' \wedge (b \wedge c) &= (\widetilde{A}' \wedge \widetilde{C}' J') \wedge (\widetilde{A}J) \\ &= \langle\, (\widetilde{A}' \wedge \widetilde{C}' J') \wedge (\widetilde{A}J)\, \rangle_3 \\ &= \langle\, (\widetilde{A}' \wedge \widetilde{C}' J')(\widetilde{A}J)\, \rangle_3 \\ &= \langle\, \widetilde{A}' \wedge \widetilde{C}' \widetilde{A}JJ'\, \rangle_3 \\ &= \langle\, \widetilde{A}' \wedge \widetilde{C}' \widetilde{A} \rangle_3 JJ' \\ &= (\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})JJ' \,. \end{aligned} \tag{185}$$

So, let's present the other three products on the RHS of (182).

$$a \wedge c' \wedge a' = (\widetilde{B} \wedge \widetilde{C} \wedge \widetilde{B}')JJ' \,, \tag{186}$$

$$b \wedge b' \wedge c' = (\widetilde{A} \wedge \widetilde{C} \wedge \widetilde{A}')JJ' \,, \tag{187}$$

$$c \wedge a \wedge a' = (\widetilde{B} \wedge \widetilde{C}' \wedge \widetilde{B}')JJ' \,. \tag{188}$$

Now,

$$\langle\, a \wedge a' \;\, b \wedge b' \;\, c \wedge c'\, \rangle = \langle\, PQR\, \rangle \,. \tag{189}$$

Then

$$\begin{aligned} \langle\, a \wedge a' \;\, b \wedge b' \;\, c \wedge c'\, \rangle &= [(\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})JJ'] \cdot [(\widetilde{B} \wedge \widetilde{C} \wedge \widetilde{B}')JJ'] \\ &\quad - [(\widetilde{A} \wedge \widetilde{C} \wedge \widetilde{A}')JJ'] \cdot [(\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})JJ'] \\ &= (JJ')^2[(\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})] \cdot [(\widetilde{B} \wedge \widetilde{C} \wedge \widetilde{B}')] \\ &\quad - [(\widetilde{A} \wedge \widetilde{C} \wedge \widetilde{A}')] \cdot [(\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})] \,. \end{aligned} \tag{190}$$

This brings us to Eq. (4.13),

$$\langle\, a \wedge a' \;\, b \wedge b' \;\, c \wedge c'\, \rangle = JJ' \langle\, \widetilde{A} \wedge \widetilde{A}' \;\, \widetilde{B} \wedge \widetilde{B}' \;\, \widetilde{C} \wedge \widetilde{C}'\, \rangle \,. \tag{191}$$

Rephrasing this, in accord with the figure:

$$\begin{aligned}
\langle\, PQR\,\rangle &= JJ'\langle\,(A\vee A')^{\sim}\ (B\vee B')^{\sim}\ (C\vee C')^{\sim}\,\rangle \\
&= JJ'\langle\,\widetilde{p}\,\widetilde{q}\,\widetilde{r}\,\rangle \\
&= JJ'\langle\, pI^{-1}\, qI^{-1}\, rI^{-1}\,\rangle \\
&= JJ'I\langle\, p\, q\, r\,\rangle_3 \\
&= JJ'I\langle\, p\, q\, r\,\rangle_3 \\
&= JJ'p\cdot(q\times r) \\
&= JJ'[pqr]\,, \qquad (192)
\end{aligned}$$

where we have use the Gibbs's cross product.

Now, by our assumptions, $JJ'\neq 0$. Thus, the three lines P, Q, and R are concurrent (at D) iff $\langle\, PQR\,\rangle = 0$ iff points p, r, and q are joined on the same line iff $p\cdot(q\times r)=0$. And that finishes Desargues's Theorem.

15 Pascal's Theorem

Now, we're skipping a few pages (and a few theorems) and going to the subsection on Conics [4.3, p.44, p.18] to get to the last theorem I'll deal with this time through the paper, which is Pascal's Theorem. This takes us to page at [p.44, p.18].

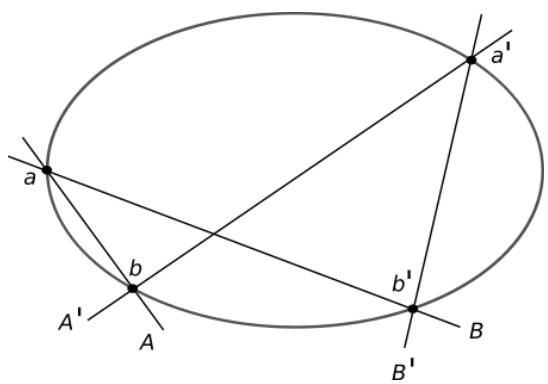

Figure 10. Pascal's Theorem: Our first task is to figure out how to 'construct' an ellipse out of our proper objects of points, lines, and planes. Remember, all the points shown on the ellipse above are the tips of vectors from the origin to the projective plane, in which our ellipse resides.

Definition: The *pencil of lines* through a point is the set of all lines through that point.

If we look at the point a on the ellipse in Fig. 10, we see two of the lines of the infinite pencil of lines that go through point a in $\mathcal{P}_2$. At this point, I want to explain the meaning of the sentence:

> The set of all lines passing through a point is called a *pencil of lines*. Every such pencil is uniquely determined by two of its lines A, B and can be represented by the expression $A + \lambda B$ where $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$. [p.44, p.18]

Before we attempt to tackle lines in the projective plane, how about we try to grasp points on a projective line.

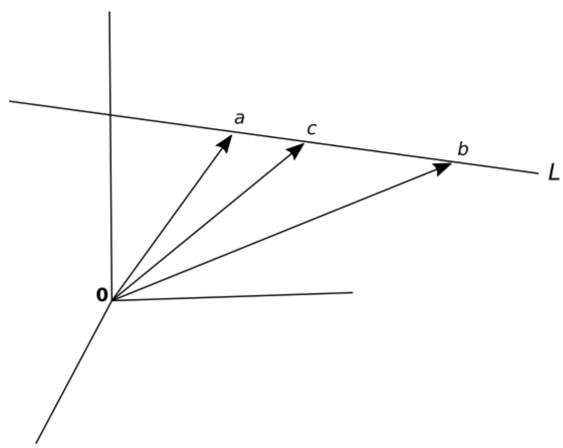

Figure 11. The line L can be considered as our $\mathcal{P}_1$ space. Question: given points a and b and real scalar λ, can we find a point c such that $ac/ab = \lambda$?

Let a and b be points on a line L and let ab be the length of line segment $\overline{ab}$. Furthermore, let ac be the length of line segment $\overline{ac}$. Can we find a point on L such that

$$\frac{ac}{ab} = \lambda\,? \tag{193}$$

Well, we can get to any point on L by first arriving at point b, say, on L and then traveling some multiple α amount of the vector $\vec{ab}$, or rather $a - b$. Hence,

$$c = b + \alpha(a - b) = \alpha a + (1 - \alpha)b\,. \tag{194}$$

The middle expression of this last equation shows us that we get find c by first arriving somewhere on L and then sliding along L some suitable amount in the direction of L, which is along the vector $a - b$. However, the RHS expression has completely changed the way we look at the problem, for it tells us that we can arrive at the point c be merely adding the right ratios of a to b. We will need a generalization of this concept to understand how to construct an ellipse out of 'lines', i.e., out a bivectors.

As for answering the question asked, subtract (194) from the equation $a = a$:

$$a - c = a - [\alpha a + (1-\alpha)b] = (1-\alpha)(a-b)\,. \tag{195}$$

Therefore,

$$\frac{ac}{ab} = \frac{a-c}{a-b} = 1 - \alpha = \lambda\,. \tag{196}$$

Lastly, we solve for α

$$\alpha = 1 - \lambda\,. \tag{197}$$

Since A and B are both lines through the point a, they can both be factored with the vector a, such as

$$A = a \wedge b\,, \tag{198}$$
$$B = a \wedge b'\,. \tag{199}$$

According to the claim, we should be able to write any other line D through a as a linear combination of A and B, such as

$$D = \alpha A + \beta B\,, \tag{200}$$

which stands for

$$D = \alpha\, a \wedge b + \beta\, a \wedge b'\,, \tag{201}$$

where neither α nor β is zero. But wait! We can only know D up to an arbitrary nonzero scale factor. Hence, we can multiply through on the RHS by any nonzero real number. So we'll multiply through by α^{-1} and set $= \lambda = \dfrac{\beta}{\alpha}$, we get

$$D = a \wedge b + \frac{\beta}{\alpha}\, a \wedge b' \tag{202a}$$
$$= a \wedge b + \lambda\, a \wedge b' = A + \lambda\, B \tag{202b}$$
$$= a \wedge (b + \lambda\, b')\,. \tag{202c}$$

Then the articles goes on to say

> Two pencils of lines $X = A + \lambda B$ and $X' = A' + \mu B'$ are said to be *projectively related* if they can be put into an ordered one-to-one correspondence such that X corresponds to X' if and only if $\lambda = \mu$. In this case, the set of intersection points of corresponding lines forms a conic. [p.44, p.18]

Let ϕ be a one-to-one linear map of points on L to points on L', then, $\phi(A) = A'$, $\phi(B) = B'$, and if x is on both L and L', then is has to be a fixed point of the mapping, i.e., $\phi(x) = x$.

So, if point x should be a point of intersection of lines L and L', i.e., x is the meet of the two lines, then

$$x \wedge L = x \wedge A + \lambda\, x \wedge B = 0\,, \tag{203a}$$

maps to the requirement that x be on L'

$$x \wedge L' = x \wedge A' + \lambda\, x \wedge B' = 0\,. \tag{203b}$$

We can eliminate λ between these two equations by first multiplying (203a) through by $(x \wedge B')$ on the right, and second by multiplying (203b) through by $(x \wedge B)$ on the left, and then taking their difference, to get

$$(x \wedge A)(x \wedge B') - (x \wedge B)(x \wedge A') = 0\,, \tag{204}$$

which is Eq. (4.18). The significance of this last equation is that it scalar part is a quadratic in x, hence is the equation for a conic. But x is also the meet of the two lines L and L', hence,

$$\begin{aligned} x &= L \vee L' \\ &= (A + \lambda B) \vee (A' + \lambda B') \\ &= A \vee A' + \lambda(A \vee B' + B \vee A') + \lambda^2 B \vee B'\,. \end{aligned} \tag{205}$$

Now, we must replace x from being a function of 'lines' to a function of 'points' (vectors). Based on Fig. 10:

$$A = a \wedge b\,, \quad B = a \wedge b'\,, \tag{206a}$$

$$A' = a' \wedge b\,, \quad B' = a' \wedge b'\,, \tag{206b}$$

$$b = A \vee A'\,, \quad b' = B \vee B'\,. \tag{206c}$$

If, for convenience, we define a new point d as,

$$d \equiv A \vee B' + B \vee A'\,, \tag{207}$$

then we can simplify (205) to

$$x = b + \lambda d + \lambda^2 b'\,. \tag{208}$$

Now, for three noncollinear points, p, q, and r, then $p \wedge q \wedge r \neq 0$, and we get the parametric equation for a nondegenerate ellipse:

$$x = p + \lambda q + \lambda^2 r\,. \tag{209}$$

We can interpet this equation from the viewpoint of the embedding space: a generic point x on some conic in $\mathcal{P}_2$ is the sum of three vectors: p, λq, and $\lambda^2 r$.

Now, if we define

$$P = q \wedge r\,, \tag{210a}$$

$$Q = r \wedge p\,, \tag{210b}$$

$$R = p \wedge q\,, \tag{210c}$$

which names the three distinct lines that can be formed out of these three distinct points: p, q, and r. Now, if we wedge (209) by p on the left, we get

$$p \wedge x = \lambda p \wedge q + \lambda^2 p \wedge r = \lambda(R - \lambda Q)\,. \tag{211a}$$

By a similar procedure, we get

$$r \wedge x = r \wedge q + \lambda r \wedge q = Q - \lambda P\,. \tag{211b}$$

Thus, (211a) and (211b) constitute "two projective pencils that generate the conic [5,§24]."[24]

If we expand (204), we get

$$(x \wedge a \wedge b)(x \wedge a' \wedge b') - (x \wedge a \wedge b')(x \wedge a' \wedge b) = 0\,. \tag{212}$$

Each of the four points a, b, a', b' individually satisfy (212).

Now, by replacing x by the point p, we get

$$(p \wedge a \wedge b)(p \wedge a' \wedge b') - (p \wedge a \wedge b')(p \wedge a' \wedge b) = 0\,. \tag{213}$$

And, since each of these pseudoscalar quantities can be replaced by its magnitude (determinant) times a unit pseudoscalar I, we then get

$$[pab]I[pa'b']I - \mu[pab']I[pa'b]I = 0\,, \tag{214}$$

where $\mu \neq 0$ and which simplifies to

$$[pab][pa'b'] - \mu[pab'][pa'b] = 0\,. \tag{215}$$

As of the time of this, my first writing on the article, I do not understand where the μ factor came from.[25]

In any case, we can solve for μ by substituting the generic point p with some other definite point that lies on the curve, namely c', yielding

$$\mu = \frac{[c'ab][c'a'b']}{[c'ab'][c'a'b]}\,. \tag{216}$$

This brings us to an obvious error in the preprint version of this equation. In the preprint version (on page 19), in the left factor of the denominator of the fraction, it reads $[cab']$, which is wrong.

We should note that μ in (216) contains five distinct points, which accounts for the five degrees of freedom for an ellipse.

[24] This source reference is to H. G. Forder's interesting book *The Calculus of Extension*. (Sub)section 24 is in Chapter III: "Applications to Projective Geometry." The book is available for reading on The **Internet Archive**.

[25] I think I'll make a guess where this μ factor came from. We have been treating objects as equivalent if they differ only by a nonzero scalar factor, and this works fine when we don't add these things together. When we do add these objects together, we may need to include a nonzero scalar factor to resynchronize various terms within a sum. Now, it might be that $\mu = 1$ or it might not.

Anyway, Eq. (215) can now be written as

$$[pab][pa'b'][c'ab'][c'a'b] - [pab'][pa'b][c'ab][c'a'b'] = 0\,. \tag{217}$$

Now, we reorder the points in the determinants to put them in standard form (except that when a p is present, it will be first):

$$[pab][pa'b'][ab'c'][a'bc'] - [pab'][pa'b][abc'][a'b'c'] = 0\,, \tag{218}$$

which is Eq. (4.19).

Thus, our conic has been decided by the five distinct points that lie on it: a, b, a', b', c'. Now, let c be an additional point on the ellipse, which gives us the relation

$$[cab][ca'b'][ab'c'][a'bc'] - [cab'][ca'b][abc'][a'b'c'] = 0\,. \tag{219}$$

On reordering the factors in the square brackets, as we did before, we get

$$[abc][a'b'c][ab'c'][a'bc'] - [ab'c][a'bc][abc'][a'b'c'] = 0\,. \tag{220}$$

On reordering the square brackets themselves, we get Eq. (4.20)

$$[abc][ab'c'][a'bc'][a'b'c] - [a'b'c'][ab'c][a'bc][abc'] = 0\,. \tag{221}$$

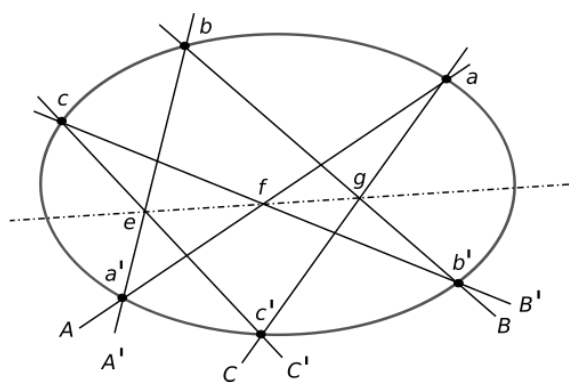

Figure 12. Pascal's Theorem with the ellipse all decked out. Our hypothesis is that $e \wedge f \wedge g = 0$, for, in that case, points e, f, and g are collinear.

So, our hypothesis is that points e, f, and g are collinear, in which case

$$e \wedge f \wedge g = 0\,. \tag{222}$$

The plan forward is to express the points e, f, and g as the meets of appropriate pairs of lines among A, A', B, B', C, C'. After that, we replace each line by the join of the pair of points on the ellipse (conic) that formed it.

Therefore, we re-express the interior points as meets of the two appropriate lines:

$$e = A' \vee C'\,, \quad f = A \vee B'\,, \quad g = C \vee B\,. \tag{223}$$

But, as we already noted, these lines through the ellipse are the joins of points that line on the ellipse, hence,

$$A = a' \wedge a\,, \quad B = b' \wedge b\,, \quad C = c' \wedge a\,, \tag{224a}$$

$$A' = a' \wedge b\,, \quad B' = b' \wedge c\,, \quad C' = c' \wedge c\,. \tag{224b}$$

Thus, (222) becomes, at first,

$$(A' \vee C') \wedge (A \vee B') \wedge (C \vee B) = 0\,. \tag{225}$$

Finally, we employ the joins to get

$$[\,(a' \wedge b) \vee (c' \wedge c)\,] \wedge [\,(a' \wedge a) \vee (b' \wedge c)\,] \wedge [\,(c' \wedge a) \vee (b' \wedge b)\,] = 0\,. \tag{226}$$

This last equation is equivalent to Eq. (4.21) in the article, although its order of presentation of the factors is different, though this is of no concern because the two corresponding expressions can only differ by a possible overall factor of a minus sign, which is irrelevant because we are only interested when the LHS expression is equal to zero.

So, how do we proceed to compute (226)? Well, the article claims that the equivalence between Equations (221) and (226) is "readily verified." Well, I don't know about that, but it is straightforward.

But first, a word about those scalars '$[xyz]$'. They are the magnitude of pseudoscalars, which in this case are trivectors. Therefore, they're also determinants. However, in the papers I wrote on my comments on Dorwart's book on incidence geometry [2], I referred to them as 'scalar products,' as they are scalars produced by the products of three vectors, namely, for the three vectors x, y, z

$$[xyz] = x \cdot (y \times z)\,. \tag{227}$$

These scalar products can be cyclically permuted:

$$[xyz] = [zxy] = [yzx]\,. \tag{228}$$

Now, in my papers on Dorwart's book, I computed the meet of two 'lines' to find their point of intersection (which I called its 'projective location') by taking the cross product of the two cross products of the two oriplanes, each of which that represent one the lines in the projective plane, in other words,

$$[abcd] \equiv (a \times b) \times (c \times d)\,. \tag{229a}$$

$$= c[d \cdot (a \times b)] - d[c \cdot (a \times b)] \tag{229b}$$

$$= c[dab] - d[cab] \tag{229c}$$

$$= c[abd] - d[abc]\,. \tag{229d}$$

Clearly, then, by the antisymmetry of the cross product,

$$\begin{aligned} [abcd] &= -[cdab]\,, \\ [abcd] &= -[bacd]\,, \\ [abcd] &= -[abdc]\,. \end{aligned} \tag{230}$$

We can convert this triple cross product into the form I prefer,

$$\begin{aligned}
[abcd] &= -[cdab] \\
&= -a[bcd] + b[acd] \\
&= b[acd] - a[bcd] \\
&= [b][acd] - [a][bcd]\,,
\end{aligned} \tag{231}$$

where I used the notation $[x] = x$, which I used in my review of Dorwart's book, but will not use here.

Transposing any two 'touching' vectors is antisymmetric:

$$[xyz] = -[xzy]\,, \quad [xyz] = -[yxz]\,. \tag{232}$$

Now, to establish the equivalence of (221) and (226), I will start with (226) and produce (221). Once that is done, going the other direction is obvious.

My first step is to convert (226) to a scalar product by multiplying both sides by the unit pseudoscalar, to get

$$[\,(a' \wedge b) \vee (c' \wedge c)\,] \cdot \big\{[\,(a' \wedge a) \vee (b' \wedge c)\,] \times [\,(c' \wedge a) \vee (b' \wedge b)\,]\big\} = 0\,. \tag{233}$$

Now, the meet of the two lines $(a' \wedge b)$ and $(c' \wedge c)$ can be written as their projective location

$$(a' \wedge b) \vee (c' \wedge c) = [a'bc'c]\,. \tag{234}$$

With this notation, (233) becomes

$$[a'bc'c] \cdot \big\{[a'ab'c] \times [c'ab'b]\big\} = 0\,. \tag{235}$$

But, how should we expand these 'projective locations'? Note that in (221) all the scalar products have one each of an a, a b, and a c, with or without a prime on it. Hence, we should expand out projective locations to be consistent with that. Then, with a small modification, we start the problem of expansion with this form:

$$[c'ca'b] \cdot \big\{[a'ab'c] \times [b'bc'a]\big\} = 0\,. \tag{236}$$

Let $\Omega \equiv [c'ca'b] \cdot \big\{[a'ab'c] \times [b'bc'a]\big\}$, then

$$\begin{aligned}
\Omega &= \big\{c[c'a'b] - c'[ca'b]\big\} \cdot \big\{(a[a'b'c] - a'[ab'c]) \times (b[b'c'a] - b'[bc'a])\big\} \\
&= \big\{c[c'a'b] - c'[ca'b]\big\} \\
&\quad \cdot \big\{([ab][a'b'c][b'c'a] - [ab'][a'b'c][bc'a] \\
&\qquad - [a'b][ab'c][b'c'a] + [a'b'][ab'c][bc'a])\big\}\,.
\end{aligned} \tag{237}$$

And now for the last expansion step:

$$\begin{aligned}
\Omega = {} & [cab][c'a'b][a'b'c][b'c'a] - [ca'b][c'a'b][a'b'c][bc'a] \\
& - [ca'b][c'a'b][ab'c][b'c'a] + [ca'b'][c'a'b][ab'c][bc'a] \\
& - [c'ab][ca'b][a'b'c][b'c'a] + [c'ab'][ca'b][a'b'c][bc'a] \\
& + [c'a'b][ca'b][ab'c][b'c'a] - [c'a'b'][ca'b'][ab'c][bc'a]\,.
\end{aligned} \tag{238}$$

Except for the first and last terms, the others cancel out in pairs, leaving us with

$$\Omega = [cab][c'a'b][a'b'c][b'c'a] - [c'a'b'][ca'b][ab'c][bc'a]\,. \tag{239}$$

This equation is the same as that in (221), after the vectors are reordered. Thus, if we set $\Omega = 0$, we get the condition that the points lie on a conic, and establishes the implication: (226) $\Rightarrow$ (221). We can reorder the vectors in the scalar products in a more lexical ordering.

$$\Omega = [abc][a'bc'][a'b'c][ab'c'] - [a'b'c'][a'bc][ab'c][abc']\,. \tag{240}$$

And this takes us as far into the article as I have time to pursue at this point.

I'll end this section with the comment that Stefanovic and Milosevic [7] took an approach similar to the approach I took here. They also found that most of the terms in the sum in (238) cancel out. They also used what I call the 'projective locations' to represent the meets of lines defined by the joins of points, which are triple cross products (p. 621).

16 Conclusion

Hopefully, this introduction presented here as my personal notes on the article will assist the determined reader to start to grasp projective geometry of the projective plane by the visualization given first in the Gibbs's vector algebra and then, more elegantly, by the geometric algebra.

References

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