Basic Introduction to the Riemann Sphere and Stereographic Projection

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Abstract

This paper introduces the Riemann sphere and shows how to map points in the unit upper-half sphere to points in the xy-plane and then back again. Basic knowledge of 3D coordinates and vectors is assumed. The discussion here is basic for the topic.

1 Introduction

The purpose of the Riemann sphere in this paper is to find an invertible map from the (upper hemi)sphere to the plane. Such a map is called a *stereographic projection.* For this paper, we won't bother to map the north pole point $\mathbf{x} = (0, 0, 1) = \mathbf{e}_3$, which would go to infinity. Also, technically, the Riemann sphere is considered to be an extension of the complex plane, but we won't use that here.

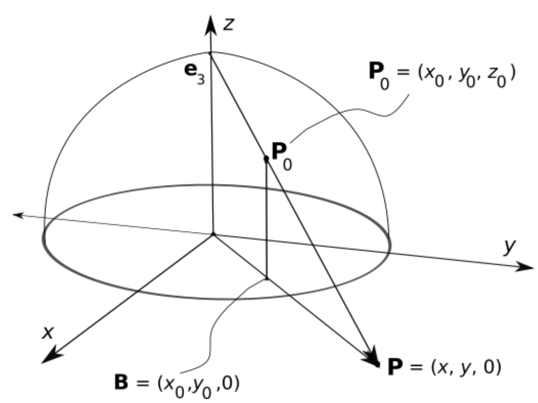

Figure 1. Depicted here is the upper half of a unit sphere. Unit vectors along the axes are $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, which are standard. The point $\mathbf{P}$ has the restriction $x^2 + y^2 \geq 1$.

But to show this correspondence, the three coordinates of the xy-plane $(x, y, 0)$ are mapped to the two coordinates of the complex plane by

$$(x, y, 0) \sim x + yi\,, \tag{1}$$

where the coordination is performed in the obvious way.

2 Mappings

Without being overly formal here in defining our maps, consider a ray emanating from the north pole of the upper hemisphere, as depicted in Figure 1. This ray will encounter the xy-plane at point $\mathbf{P} = (x, y, 0)$, passing through the sphere at point $\mathbf{P}_0 = (x_0, y_0, z_0)$.

Note:

$$\begin{cases} \mathbf{e}_3^2 & = 1\,, \\ \mathbf{e}_3 \cdot \mathbf{P} & = 0\,, \\ \mathbf{e}_3 \cdot \mathbf{P}_0 & = z_0\,, \\ \mathbf{P}_0^2 & = x_0^2 + y_0^2 + z_0^2 = 1\,. \end{cases} \tag{2}$$

Case 1.

Our first mapping problem is as follows: Given the coordinates of point $\mathbf{P}_0$, find the coordinates of point $\mathbf{P}$. We begin with the fundamental equation

$$\mathbf{P}_0 = \mathbf{P} + \lambda(\mathbf{e}_3 - \mathbf{P})\,, \tag{3}$$

which is interpreted as follows: Start at point $\mathbf{P}$ and then go λ amount in the direction $\mathbf{e}_3 - \mathbf{P}$ until you hit point $\mathbf{P}_0$. Of course, we have to derive λ by using our constraints. On dotting through by $\mathbf{e}_3$, we get that

$$\mathbf{e}_3 \cdot \mathbf{P}_0 = z_0 = \lambda\,. \tag{4}$$

Thus, (3) becomes

$$\mathbf{P}_0 = \mathbf{P} + z_0(\mathbf{e}_3 - \mathbf{P})\,, \tag{5}$$

which can be rewritten as

$$\mathbf{P}_0 - z_0\mathbf{e}_3 = \mathbf{P}(1 - z_0)\,. \tag{6}$$

From this we get

$$(x_0, y_0, 0) = (x, y, 0)(1 - z_0)\,. \tag{7}$$

Therefore,

$$\begin{cases} x = \dfrac{x_0}{1 - z_0}\,, \\ y = \dfrac{y_0}{1 - z_0}\,. \end{cases} \tag{8}$$

Case 2.

Our next mapping problem is as follows: Given the coordinates of point $\mathbf{P}$, find the coordinates of point $\mathbf{P}_0$. Now, one might think that the inversion of the coordinates as given to us in (8) would be a simple matter, but I did not find that to be the case. If you get started on the wrong foot, you could go in circles.

Anyway, let's start with

$$\mathbf{P}_0^2 = x_0^2 + y_0^2 + z_0^2 = 1\,, \tag{9}$$

and solve for z_0^2:

$$z_0^2 = 1 - x_0^2 - y_0^2\,, \tag{10}$$

which can be written as

$$z_0^2 = 1 - (1 - z_0)^2(x^2 + y^2) = 1 - (1 - z_0)^2\alpha\,, \tag{11}$$

where $\alpha \equiv x^2 + y^2$. On solving (11) for z_0, we get

$$z_0 = \frac{\alpha - 1}{\alpha + 1} = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\,. \tag{12}$$

From this we get that

$$1 - z_0 = \frac{2}{1 + x^2 + y^2}\,. \tag{13}$$

So, on inverting (8), we get, together with (12),

$$\begin{cases} x_0 = \dfrac{2x}{1 + x^2 + y^2}\,, \\ y_0 = \dfrac{2y}{1 + x^2 + y^2}\,, \\ z_0 = \dfrac{x^2 + y^2 - 1}{x^2 + y^2 + 1}. \end{cases} \tag{14}$$