Notes on *Universal Geometric Algebra*

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Abstract

This paper contains my personal notes on the paper *Universal Geometric Algebra*.[1] My comments are meant 1) to clarify certain parts of the exposition (especially for readers, like myself, who are not experts in projective geometry), 2) to fill-in some of the steps in the mathematical derivations, and 3) to report on a few mistakes that remain in the preprint version of the paper. As a word of warning, this paper will make no attempt to teach the fundamentals of geometric/Clifford algebra, though it will spend some time enhancing the discussion on it presented in the paper.

1 Introduction

This paper is the second of a series of three papers on projective geometry (and linear algebra) papers written by D. Hestenes and his coauthors. The first paper was *Projective Geometry with Clifford Algebra*,[2] These papers were published in the late 1980s and early 1990s. If projective geometry is new to the reader, I have prepared an introduction to it in my notes on the previous paper, *Projective Geometry with Clifford Algebra.* For this paper, I assume that the reader is familiar enough with projective geometry to follow these notes without too much explanation on it by me.

At the moment, I have only access to the preprint version of the article. Therefore, the references I give will be by page number of that version.

It is not my purpose to present a full introduction to geometric algebra in these notes. However, I will try to flesh-out some of the steps to the equations that have been left to the reader to provide.

[1] D. Hestenes, *Universal geometric algebra*, Quarterly Jour. of Pure and Applied Mathematics, Simon Stevin 62, 253–274, (September – December, 1988).

[2] D. Hestenes, R. Ziegler, *Projective Geometry with Clifford Algebra*, *Acta Appl. Math.* Vol. 23, 25–63 (1991).

2 The unreasonable effectiveness of projective geometry

The projective plane $\mathcal{P}_2$ is a plane embedded into a vector space of three dimensions. This embedding is arbitrary, so long as the plane $\mathcal{P}_2$ does not contain the origin. The points in this plane are vectors from the origin of the embedding space to the point in the plane. But wait. We don't care about the length of this vector, so can rescale the vector by any nonzero amount and the new vector is just as good to represent the point as the original vector.

Line segments (or rather lines) in $\mathcal{P}_2$ are the intersections of some oriplane with the plane $\mathcal{P}_2$. An **oriplane** is a plane in the embedding vector space that contains the origin. An **oriline** is a line in the embedding vector space that contains the origin.[3] Basically, there are two related ways to algebraically represent a given oriplane. One is by its normal vector and the other is by a bivector that has the same direction in space as does the oriplane. The usual way to do this is to define a line in $\mathcal{P}_2$ as the join of two points in the plane, say a and b. Thus, in symbolic form, the join of a and b is represented as $a \wedge b$. Fortunately, $a \wedge b$ is the bivector of the oriplane, and it functions as a typical bivector in the geometric algebra G_3. Thus, the bivector $a \wedge b$ represents an oriplane, and the oriplane interescts $\mathcal{P}_2$ in the line a join b.

The magic here is that the objects that represent points and lines in $\mathcal{P}_2$ are in a dimension of one step higher than in $\mathcal{P}_2$. And this facilitates computation.

3 Section 1 of the paper: Geometric Algebra

A multivector is the arbitrary sum of graded parts of the algebra G_n in n dimensions. Therefore, if we represent our arbitrary multivector by M, then

$$M = \sum_{k=0}^{n} M_k = \sum_{k=0}^{n} \langle\, M\, \rangle_k\,, \tag{1}$$

where M_0 is the scalar part, M_1 is the vector part, all the way up to M_n which is the pseudoscalar part. The pseudoscalar part of a multivector has properties that generally depend on n, such as whether or not it commutes with some other element of the algebra. There is always a unit pseudoscalar, and all other pseudoscalars are scalar multiples of it. In 3-D space (i.e., G_3) the pseudoscalar commutes with all the elements of G_3, which is a highly convenient property.

Let's begin with a k-blade $M_k = a_1 \wedge \ldots \wedge a_k$, where the a_i's are vectors. The dagger operator reverses the ordering of the vectors in the wedge product, thus

$$M_k^\dagger = a_k \wedge \ldots \wedge a_1\,. \tag{2}$$

So, what's the relation of $M_k^\dagger$ to M_k (Eq. (5) [p.3])? To help us answer this, we

[3] The plane $\mathcal{P}_2$ is any plane in the embedding space that is not an oriplane.

begin with a lemma: the sum of integers from 1 to n is given as

$$\sum_{j=1}^{n} = 1 + 2 + \cdot + n = \frac{(n+1)n}{2} \,. \tag{3}$$

Hence,

$$\begin{aligned}
\langle\, M^{\dagger} \,\rangle_k = \langle\, M \,\rangle_k^{\dagger} = M_k^{\dagger} &= a_k \wedge \ldots \wedge a_1 \\
&= (-1)^{k-1} a_1 \wedge a_k \wedge \ldots \wedge a_2 \\
&= (-1)^{(k-1)+(k-2)} a_1 \wedge a_2 \wedge a_k \wedge \ldots \wedge a_3 \\
&= \cdots \\
&= (-1)^{(k-1)+(k-2)+\cdots+1} a_1 \wedge a_2 \cdots \wedge a_k \\
&= (-1)^{k(k-1)/2} M_k \\
&= (-1)^{k(k-1)/2} \langle\, M \,\rangle_k \,.
\end{aligned} \tag{4}$$

Next, we have Eq. (6) [p.3]. Let $A = \langle\, A \,\rangle_r$ and $B = \langle\, B \,\rangle_s$ both be blades,[4] then

$$A \wedge B \equiv \langle\, AB \,\rangle_{r+s} = (-1)^{rs} B \wedge A \,. \tag{5}$$

So, think of A as a blade of r vectors wedged together, and B as a blade of s vectors wedged together. To change the order of A and B in the wedge product, we must make s transpositions for each of r vectors in A. That makes for a total of rs transpositions, and each transposition gives a factor of (-1).

Next, we have Eq. (7) [p.3].

$$A \cdot B = \langle\, AB \,\rangle_{r-s} = (-1)^{s(r-s)} B \cdot A \qquad \text{for } r \geq s \,. \tag{6}$$

So, for $r \geq s$, [note: $(-1)^{r(r-1)} = 1$ for all integers r]

$$\begin{aligned}
A \cdot B &= \langle\, AB \,\rangle_{r-s} \\
&= \langle\, AB \,\rangle_{r-s}^{\dagger\dagger} \\
&= \langle\, B^{\dagger} A^{\dagger} \,\rangle_{r-s}^{\dagger} \\
&= (-1)^{(r-s)(r-s-1)/2} \langle\, B^{\dagger} A^{\dagger} \,\rangle_{r-s} \\
&= (-1)^{(r-s)(r-s-1)/2} (-1)^{s(s-1)/2} (-1)^{r(r-1)/2} \langle\, BA \,\rangle_{r-s} \\
&= (-1)^{r^2+s^2-r-rs} \langle\, BA \,\rangle_{r-s} \\
&= (-1)^{r(r-1)+s(s-r)} \langle\, BA \,\rangle_{r-s} \\
&= (-1)^{s(s-r)} \langle\, BA \,\rangle_{r-s} \\
&= (-1)^{s(r-s)} \langle\, BA \,\rangle_{r-s} \\
&= (-1)^{s(r-s)} B \cdot A \,,
\end{aligned} \tag{7}$$

[4] Blades are k-vectors that can be factored into one term of k vectors, wedged together.

where we used the fact that for any integer n, $(-1)^n = (-1)^{-n}$.

Now we're at [p.4, Eq. (9)].
Here we introduce the duals of r-blades $A = \langle A \rangle_r$:

$$\widetilde{A} \equiv AI^{-1} = A \cdot I^{-1} = (-1)^{r(n-r)} I^{-1} A\,. \tag{8}$$

Hint: Use (6) with $r \to n$ and $s \to r$.

For Eq. (10) of the preprint, with $A = \langle A \rangle_r$ and $B = \langle B \rangle_s$,

$$A \cdot (BI) = (-1)^{s(n-s)} (AI) \cdot B\,. \tag{9}$$

Proof:

$$\begin{aligned} A \cdot (BI) &= (-1)^{s(n-s)} A \cdot (IB) \\ &= (-1)^{s(n-s)} \langle AIB \rangle_{n-r-s} \\ &= (-1)^{s(n-s)} \langle (AI) \cdot B \rangle_{n-r-s} \\ &= (-1)^{s(n-s)} (AI) \cdot B\,. \end{aligned} \tag{10}$$

Next, we have Eq. (11) [p.4] of the preprint:

$$(A \wedge B)^{\sim} = A \cdot \widetilde{B} = (-1)^{s(n-s)} \widetilde{A} \cdot B\,. \tag{11}$$

So,

$$\begin{aligned} (A \wedge B)^{\sim} &= (A \wedge B) I^{-1} = \langle AB \rangle_{r+s} I^{-1} \\ &= \langle ABI^{-1} \rangle_{n-(r+s)} \\ &= (-1)^{s(n-s)} \langle AI^{-1} B \rangle_{n-(r+s)} \\ &= (-1)^{s(n-s)} \langle \widetilde{A} B \rangle_{(n-r)-s} \\ &= (-1)^{s(n-s)} \widetilde{A} \cdot B\,. \end{aligned} \tag{12}$$

And then,

$$\begin{aligned} (A \wedge B)^{\sim} &= \langle ABI^{-1} \rangle_{n-(r+s)} \\ &= \langle A(BI^{-1}) \rangle_{(n-s)-r} \\ &= A \cdot \widetilde{B}\,. \end{aligned} \tag{13}$$

4 Section 2 of the paper: The Algebra of Subspaces

We begin with a vector space of n dimensions $\mathcal{V}^n$. It's crucial to distinguish between a subspace of a vector space and merely a subset of it. For example, every line within this vector space is a subset of the space, but only those lines through the origin are subspaces. Why? Because a subspace is a vector space in its own right, and as such, it must contain the origin, which is the zero vector. For examples, the origin of a vector space by itself is a trivial subspace. Subspaces of one dimension are the lines through the origin; the planes through the origin are the subspaces of two dimensions, etc. So, we can in geometric algebra a subspace of rank r by an r-blade $A = \langle A_r \rangle$, where the base of each vector in A is at the origin.

With that basic understanding, then we can characterize the set of all points in the subspace spanned by the r vectors in A by (Eq. (12) of the preprint)

$$\{x : x \wedge A = 0\}\,, \tag{14}$$

where we understand that the tip of vector x corresponds to the point of the subset.

Next we move to Grassmann's regressive product. [Eq. (13) of the preprint]

$$A \vee B \equiv \widetilde{A} \cdot B\,. \tag{15}$$

For Eq. (14),

$$(A \vee B)^{\sim} = \widetilde{A} \wedge \widetilde{B}\,. \tag{16}$$

With $A = \langle A \rangle_r$ and $B = \langle B \rangle_s$, and $r + s > n$

$$\begin{aligned}
(A \vee B)^{\sim} &= A \vee B I^{-1} \\
&= \widetilde{A} \cdot B I^{-1} \\
&= \langle \widetilde{A} B \rangle_{(n-r)-s} I^{-1} \\
&= \langle \widetilde{A} B I^{-1} \rangle_{(n-r)+(n-s)} \\
&= \langle \widetilde{A} \widetilde{B} \rangle_{(n-r)+(n-s)} \\
&= \widetilde{A} \wedge \widetilde{B}\,.
\end{aligned} \tag{17}$$

Next we have (15) from the preprint: From (11) above

$$(A \wedge x)^{\sim} = (-1)^{n-1} \widetilde{A} \cdot x\,. \tag{18}$$

Hence,

$$\begin{aligned}
A \wedge x &= (-1)^{n-1} (\widetilde{A} \cdot x) I^{-1} \\
&= (-1)^{n-1} (\widetilde{A} \cdot x) I^{\dagger} \\
&= (-1)^{n-1} (-1)^{n(n-1)/2} (\widetilde{A} \cdot x) I\,.
\end{aligned} \tag{19}$$

Then

$$(A \wedge x) I^{-1} = (-1)^{n-1} (-1)^{n(n-1)/2} \widetilde{A} \cdot x\,. \tag{20}$$

Now, reverse A and x on the LHS to get

$$(-1)^{r(r-1)/2} (x \wedge A) I^{-1} = (-1)^{n-1} (-1)^{n(n-1)/2} \widetilde{A} \cdot x\,. \tag{21}$$

If we set $x \wedge A = 0$, then

$$0 = (x \wedge A) I^{-1} = \widetilde{A} \cdot x = a \cdot x\,. \tag{22}$$

5 Section 3 of the paper: Projective Geometry

This section is an overview of the paper *Projective Geometry with Clifford Algebra*, which is available on David Hestenes's website. Also, I did my own set of notes on this paper, which is available on this website.

The important takeaway from this section is that two lines A and B in $\mathcal{P}_2$ will intersect if their oriplane bivectors can be factored to reveal a common vector. Let's look closer at this. Say we can factor the 2-blades A and B as

$$A = a \wedge d \qquad B = b \wedge d \,. \tag{23}$$

(If you think about it, this is necessarily true for oriplanes in 3-D space, because the oriplanes themselves must intersect in a common line. But projective geometry can be done in higher dimensional spaces, as well.) Anyway, in this case of (23), $A \wedge B = 0$. As a consequence,

$$A \vee B = d \,. \tag{24}$$

Because I remember well how confusing this stuff was to me when I first started to learn it, I'll state what (24) means in both $\mathcal{P}_2$ and in its embedding space G_3. In G_3, it means that the two oriplanes corresponding to the two bivectors (2-blades) A and B meet in the oriline (vector) d. But in $\mathcal{P}_2$, those oriplanes intersect $\mathcal{P}_2$ in 'lines' A and B, and the vector d 'intersects' $\mathcal{P}_2$ at 'point' d.

So, for arbitrary point d in $\mathcal{P}_2$ to be concurrent with arbitrary line C in $\mathcal{P}_2$,

$$d \wedge C = 0 \,. \tag{25}$$

Why is this? Because then the oriline represented by vector d is in the oriplane represented by 2-blade C.

So, if 'point' d is itself the meet of two other lines A and B in $\mathcal{P}_2$, then we have Eq. (20) in the preprint:

$$d \wedge C = (A \vee B) \wedge C = A \wedge (B \vee C) = 0 \,. \tag{26}$$

Okay, so how is it true that $A \wedge (B \vee C) = 0$? Well, what we are investigating is the algebraic condition that three lines in $\mathcal{P}_2$ meet at a single point d. But there is nothing special about how we intially pair off the two lines A and B. We could have, instead, paired lines B and C first to get $d = B \vee C$, in which case we would have

$$A \wedge d = 0 \,, \tag{27}$$

yielding

$$A \wedge (B \vee C) = 0 \,. \tag{28}$$

6 Section 4 of the paper: Linear Algebra

This short overview of linear algebra begins with the standard linear transformations f from an n-dimensional vector space V_n to itself, namely

$$f : x \to f(x)\,. \tag{29}$$

However, corresponding to V_n is the entire geometric algebra G_n. How do all of its elements transform under f? Well, not at all really, since f is only defined on the vectors of G_n. However, we can invent a generalization of f that generalizes f in two ways.

First, we invent a symbol for this generalization, namely, $\underline{f}$.

$$\underline{f} : x \to \underline{f}(x) = f(x)\,. \tag{30}$$

So, $\underline{f}$ treats the vectors of G_n exactly as does f. Now, in the abstract sense of a vector space, $\underline{f}$ treats the many elements of G_n as 'vectors' of a linear space. Therefore, for a generic multivector

$$M = M_0 + M_1 + \cdots + M_n\,, \tag{31}$$

then

$$\underline{f}(M) = \underline{f}(M_0) + \underline{f}(M_1) + \cdots + \underline{f}(M_n)\,. \tag{32}$$

Further, $\underline{f}$ commutes with scalars, such as α:

$$\underline{f}(\alpha M_k) = \alpha\,\underline{f}(M_k)\,. \tag{33}$$

This is not a necessary requirement of $\underline{f}$, but it makes sense, since we would have to demand it to be true when $\underline{f}$ acts on vectors.

We haven't yet given $\underline{f}$ a name. We'll call it the *outermorphism* operator for simplicity.[5]

Now, if what has been presented thus far is all there is to say about the outermorphism, it wouldn't be of much use to us. The property of it of most value is how it gets its name, namely by operating on blades, thusly. For the r-blade $A = a_1 \wedge a_2 \wedge \cdots \wedge a_r$, where, of course, the a_i's are vectors,

$$\underline{f}(A_k) \equiv \underline{f}(a_1) \wedge \underline{f}(a_2) \wedge \cdots \wedge \underline{f}(a_r)\,. \tag{34}$$

We have one last detail to deal with before our definition of the outermorphism is complete. How does it map scalars?[6] Let's begin with the easier question of how it maps the unity 1. Now, it's an axiom of geometric algebra that for any k-blade B_k,

$$1B_k = 1 \wedge B_k\,, \tag{35}$$

then

$$\underline{f}(B_k) = \underline{f}(1) \wedge \underline{f}(B_k)\,, \tag{36}$$

[5] For a more rigorous treatment of this subject, see the book by Hestenes and Sobczyk, *Clifford Algebra to Geometric Calculus*.

[6] This has already been implied in (33).

but this only makes sense if $\underline{f}(1) = 1$, in which case

$$\underline{f}(1) \wedge \underline{f}(B_k) = 1 \wedge \underline{f}(B_k) = 1\,\underline{f}(B_k) = \underline{f}(B_k)\,. \tag{37}$$

Now, since $\underline{f}$ fixes the unity of G_n, it also fixes all positive integers:

$$\underline{f}(n) = \underline{f}(1 + 1 + \cdots + 1) = \underline{f}(1) + \underline{f}(1) + \cdots + \underline{f}(1) = 1 + 1 + \cdots + 1 = n\,.$$

So, the logical generalization of this is to insist that the outermorphism fixes all scalar terms. Let α be a scalar (real number), then,

$$\underline{f}(\alpha) = \alpha\,. \tag{38}$$

Thus, we arrive at Eq. (27) of the preprint:

$$\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B)\,, \tag{39}$$

where A and B are blades.

Well, what about the other end of the blade 'spectrum', the pseudoscalar of the space? We have no reason to insist that the outermorphism fix the pseudoscalar, therefore, for pseudoscalar I,

$$\underline{f}(I) = \beta I\,, \tag{40}$$

where β is some scalar value. It turns out that $\beta = \det f$, hence

$$\underline{f}(I) = (\det f) I\,. \tag{41}$$

Nonsingular linear transformations can be interpreted as collineations, so that points are mapped to points and lines are mapped to lines. Proof: Say that $\underline{f}(x) = x'$ and $\underline{f}(A) = A'$. Remember that the condition that in $\mathcal{P}_2$ that point x lie on line A is that

$$x \wedge A = 0\,. \tag{42}$$

But we can map this using the outermorphism $\underline{f}$ to

$$0 = \underline{f}(x \wedge A) = \underline{f}(x) \wedge \underline{f}(A) = x' \wedge A'\,. \tag{43}$$

Therefore, when $\underline{f}(I) \neq 0$ then x' is in the space of A'.

Of course we who have studied linear algebra know that there exists a mapping known as the *transpose* of a matrix, which in our scheme is the transpose (or *adjoint*) of a linear transformation f, denoted $\bar{f}$, which is defined implicitly here as

$$\langle\, M \bar{f} N \,\rangle \equiv \langle\, N \underline{f} M \,\rangle\,, \tag{44}$$

which holds for all multivectors M, N in G_n. Incidentally,

$$\langle\, M \bar{f} N \,\rangle = \langle\, (\bar{f} N) M \,\rangle\,, \tag{45}$$

which holds because of equality under cyclic permutation of the geometric products of multivectors in a scalar-grade selection. In other words, for multivectors A amd B,

$$\langle AB \rangle = \langle BA \rangle \,. \tag{46}$$

Next in the preprint is the stated relation between the inner product and the adjoint:

$$A \cdot (\bar{f}\, B) = \bar{f}\,[(\underline{f} A) \cdot B]\,, \tag{47}$$

where step $A \le$ step B.[7] This leads to

$$\underline{f}^{-1} A = \frac{\bar{f}\,(AI) I^{-1}}{\det f} \,. \tag{48}$$

There are certain things we can prove about $\bar{f}$ without too much trouble. For example, we can show that

$$\bar{f}\,(I) = (\det f) I \,. \tag{49}$$

Go back to Eq. (44) and let $M = N = I$, then

$$\langle I \bar{f}\, I \rangle = \langle I \underline{f} I \rangle = (\det f) I^2 \,. \tag{50}$$

The only way to make this work is to set $\bar{f}\, I = \alpha I$ where α is a scalar to be determined.

$$\langle I \bar{f}\, I \rangle = \alpha \langle II \rangle = (\det f) I^2 \,, \tag{51}$$

from which we conclude that $\bar{f}\, I = (\det f) I$.

We're now ready to show that (48) follows from (47). First, reset (47) as

$$C \cdot (\bar{f}\, B) = \bar{f}\,[(\underline{f} C) \cdot B] \,. \tag{52}$$

Now, replace B by I:

$$C \cdot (\bar{f}\, I) = \bar{f}\,[(\underline{f} C) \cdot I]\,, \tag{53}$$

which simplifies to

$$(\det f) CI = \bar{f}\,[(\underline{f} C) I] \,. \tag{54}$$

With just a little alteration we get that

$$C = \frac{\bar{f}\,[\underline{f}(C) I] \cdot I^{-1}}{\det f} \,. \tag{55}$$

Finally, on the assumption that $\underline{f}^{-1} A$ exists, let $C = \underline{f}^{-1} A$, then

$$\underline{f}^{-1} A = \frac{\bar{f}\,(A \cdot I) I^{-1}}{\det f} \,, \tag{56}$$

which simplifies to (48).

[7] This equation can be found as Eq. 1.14a on page 69 of CAGC [3].

This brings us now to Eq. (33a) in the preprint:

$$(\underline{f}A) \vee (\underline{f}B) = (\det f)\, \underline{f}(A \vee B)\,, \tag{57}$$

with proof as follows (provided in *Design of Linear Algebra and Geometry* [p.70, p.5])

$$\begin{aligned}(\underline{f}A) \vee (\underline{f}B) &= [\,(\underline{f}A)I^{-1}] \cdot (\underline{f}B) \\ &= \underline{f}[\,(\bar{f}\,[\,(\underline{f}A)I^{-1}\,]) \cdot B\,] \\ &= \underline{f}[\,\widetilde{A} \cdot B\,](\det f) \\ &= (\det f)\, \underline{f}(A \vee B)\,,\end{aligned} \tag{58}$$

where

$$A \vee B = \widetilde{A} \cdot B \quad \text{for } r + s \geq n\,. \tag{59}$$

To go from line two to line three in the last proof, one can use the results given in the book *Clifford Algebra to Geometric Calculus*, Hestenes and Sobczyk, Reidel, 1984, 1987, pg. 69, Eqs. (1.14a,b):

$$A_r \cdot \bar{f}\,(B_s) = \bar{f}\,[\,\underline{f}(A_r) \cdot B_s\,] \qquad \text{for } r \leq s\,, \tag{60}$$

$$\underline{f}(A_r) \cdot B_s = \underline{f}[\,A_r \cdot \bar{f}\,(B_s)\,] \qquad \text{for } r \geq s\,. \tag{61}$$

On turning (58) around and ignoring the factor of $(\det f)$, we have

$$\underline{f}(A \vee B) = (\underline{f}A) \vee (\underline{f}B)\,. \tag{62}$$

Substituting on the LHS yields

$$\underline{f}(\widetilde{A} \cdot B) = (\underline{f}A) \vee (\underline{f}B)\,. \tag{63}$$

Substituting on the RHS yields

$$\underline{f}(\widetilde{A} \cdot B) = (\underline{f}A)^{\sim} \cdot (\underline{f}B)\,. \tag{64}$$

Setting $A' = \underline{f}A$ and $B' = \underline{f}B$, we are at last at (33b) in the preprint article,

$$\underline{f}(\widetilde{A} \cdot B) = \widetilde{A}' \cdot B'\,. \tag{65}$$

7 Section 5 of the paper: Projective Split and Cross Ratio

Earlier, we saw the points of $\mathcal{P}_2$ as represented by vectors in V_3. Now we'll generalize this to $\mathcal{P}_n$ (V_n) as embedded in the vector space V_{n+1}.

We'll construct an algebraic relationship between V_{n+1} and V_n. We now define the set of 'vectors'

$$V_n = \{x \wedge \mathbf{e}_0 \,|\, x \in V_{n+1}\}\,, \tag{66}$$

which is Eq. (34) in the preprint paper.

If you take the geometric algebra of this, G_n, you get first the bivectors of V_{n+1}, of course. If you take inner products of bivectors, you get scalars. If you combine the bivectors to make higher-graded objects, you get 4-vectors, 6-vectors, etc. In other words, all that taken together gives us the set of even elements of G_{n+1}. But does this set have an algebraic structure? It does. Since the product of any two even elements of G_{n+1} is another even element of G_{n+1}, then the set constitutes the even subalgebra of G_{n+1}.

But what about the set of 'vectors' we defined in (66)? Is this really a 'vector' space? It certainly is under the usual definition of a vector space. It has a zero vector, namely $\mathbf{e}_0 \wedge \mathbf{e}_0 = 0$. It's closed under addition of 'vectors' and under scalar multiplication, etc.

But wait! If V_n as defined in (66) is a legitimate vector space, then it should have its own geometric algebra G_n, right? Right.

Anyway, back to analyzing a typical element of V_n, that being $x \wedge \mathbf{e}_0$. Let's think about this in $\mathcal{P}_2$, which is the projective plane in V_3. We said that if a and b are any two distinct points in $\mathcal{P}_2$, that we can represent the join of these points as $a \wedge b$. This join is a line in $\mathcal{P}_2$ containing points a and b. Furthermore, $a \wedge b$ is a 2-blade in G_3.

One way to think of $\mathcal{P}_2$ that contains the point $\mathbf{e}_0$ is that it is the set of all points in $\mathcal{P}_2$ whose joins with $\mathbf{e}_0$ are orthogonal to $\mathbf{e}_0$. The text claims that $x \wedge \mathbf{e}_0$ is a linear map from V_{n+1} to V_n. Let's investigate this a bit more formally. Let L be a map from V_{n+1} to V_n,

$$L : V_{n+1} \to V_n \quad \text{given by} \quad L(x) = x \wedge \mathbf{e}_0 \,. \tag{67}$$

Show that this mapping is linear. Let α be a scalar, then,

$$L(\alpha x) = (\alpha x) \wedge \mathbf{e}_0 = \alpha(x \wedge \mathbf{e}_0) = \alpha L(x) \,. \tag{68}$$

So, it treats scalars properly. What about vector addition?

$$L(x + y) = (x + y) \wedge \mathbf{e}_0 = x \wedge \mathbf{e}_0 + y \wedge \mathbf{e}_0 = L(x) + L(y) \,. \tag{69}$$

And it treats vector addition properly, hence, it's a linear map.

We can give the elements of V_n a cosmetic upgrade by letting $x_0 = x \cdot \mathbf{e}_0 \in \mathbb{R}$ and $\mathbf{x} \equiv x \wedge \mathbf{e}_0 / x \cdot \mathbf{e}_0$ for each $x \in V_{n+1}$, then

$$x\mathbf{e}_0 = x \cdot \mathbf{e}_0 + x \wedge \mathbf{e}_0 = x_0(1 + \mathbf{x}) \,, \tag{70}$$

which is Eq. (36) in the preprint paper.

Lemma:

$$\mathbf{e}_0 x = x_0(1 - \mathbf{x}) \,. \tag{71}$$

Proof:

$$\mathbf{e}_0 x = (x\mathbf{e}_0)^\dagger = [\, x_0(1 + x \wedge \mathbf{e}_0)\,]^\dagger = x_0(1 - x \wedge \mathbf{e}_0) = x_0(1 - \mathbf{x}) \,. \tag{72}$$

We'll now prove Eq. (37), which is

$$a \wedge b = a_0 b_0 (\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}) = a_0 b_0 (\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,, \tag{73}$$

where $\mathbf{e}_0^2 = 1$, $a\mathbf{e}_0 = a_0(1 + \mathbf{a})$, and $b\mathbf{e}_0 = b_0(1 + \mathbf{b})$.

So, for the proof:

$$\begin{aligned}
a \wedge b &= \tfrac{1}{2}(ab - ba) \\
&= \tfrac{1}{2}(a\mathbf{e}_0\mathbf{e}_0 b - b\mathbf{e}_0\mathbf{e}_0 a) \\
&= \tfrac{1}{2}[\,(a\mathbf{e}_0)(\mathbf{e}_0 b) - (b\mathbf{e}_0)(\mathbf{e}_0 a)\,] \\
&= \tfrac{1}{2}[\,a_0 b_0 (1 + \mathbf{a})(1 - \mathbf{b}) - a_0 b_0 (1 + \mathbf{b})(1 - \mathbf{a})\,] \\
&= a_0 b_0 [\,\mathbf{a} - \mathbf{b} + \tfrac{1}{2}(\mathbf{ba} - \mathbf{ba})\,] \\
&= a_0 b_0 [\,\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}\,]\,.
\end{aligned} \tag{74}$$

If we let $\mathbf{u} \equiv \mathbf{a} - \mathbf{b}$ and $\mathbf{M} \equiv \mathbf{a} \wedge \mathbf{u} = \mathbf{b} \wedge \mathbf{a}$, this last result can be written as

$$a \wedge b = a_0 b_0 (\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}) = a_0 b_0 (\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,. \tag{75}$$

Lemma:

$$\mathbf{a} \wedge \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{b} \wedge \mathbf{a} = -\mathbf{c} \cdot \mathbf{a} \wedge \mathbf{b}\,, \tag{76}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors.

Proof:

$$\langle\, \mathbf{abc} \,\rangle_1 = \langle\, \mathbf{abc} \,\rangle_1^\dagger = \langle\, \mathbf{cba} \,\rangle_1\,.$$

Expanding both sides,

$$\langle\, \mathbf{a} \cdot \mathbf{bc} + \mathbf{a} \wedge \mathbf{bc} \,\rangle_1 = \langle\, \mathbf{cb} \cdot \mathbf{a} + \mathbf{cb} \wedge \mathbf{a} \,\rangle_1\,.$$

On dropping a term, gives

$$\langle\, \mathbf{a} \wedge \mathbf{bc} \,\rangle_1 = \langle\, \mathbf{cb} \wedge \mathbf{a} \,\rangle_1\,.$$

Hence,

$$\mathbf{a} \wedge \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{b} \wedge \mathbf{a} = -\mathbf{c} \cdot \mathbf{a} \wedge \mathbf{b}\,, \tag{77}$$

Now, on to Eq. (38), which is a bit more involved.

$$x \wedge a \wedge b = x_0 a_0 b_0 [\,(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0 = 0\,. \tag{78}$$

The reason this quantity is zero is by design, since we are looking for all x that lie in the plane described by the 2-blade $a \wedge b$. So, we begin:

Let $B \equiv a \wedge b$, then $x \wedge a \wedge b$ can be expressed as $x \wedge B$. Hence,

$$x \wedge B = \tfrac{1}{2}(xB + Bx)\,. \tag{79}$$

Now, it's time to be a little bit tricky in how we introduce the projective split. We'll start by multiplication on the right by $\mathbf{e}_0$:

$$2x \wedge B\mathbf{e}_0 = xB\mathbf{e}_0 + Bx\mathbf{e}_0\,. \tag{80}$$

We already have an expression for $B = a \wedge b$ in (75), therefore,

$$\begin{aligned} 2x \wedge B\mathbf{e}_0 = &\ xB\mathbf{e}_0 + Bx\mathbf{e}_0 \\ &= x[\,a_0b_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,]\mathbf{e}_0 + [\,a_0b_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\,]x\mathbf{e}_0\,. \end{aligned} \tag{81}$$

Let $\Omega = 2x \wedge B\mathbf{e}_0/x_0a_0b_0$, then (with $\mathbf{e}_0^2 = 1$)

$$\begin{aligned} \Omega x_0 &= x[\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0 + [\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]x\mathbf{e}_0 \\ &= x\mathbf{e}_0\mathbf{e}_0[\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0 + [\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]x\mathbf{e}_0 \\ &= x_0(1 + \mathbf{x})\{\mathbf{e}_0[\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0\} + [\,\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,]x_0(1 + \mathbf{x})\,. \end{aligned} \tag{82}$$

Therefore, some simplification yields

$$\Omega = (1 + \mathbf{x})\{\mathbf{e}_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\mathbf{e}_0\} + (\mathbf{u} + \mathbf{a} \wedge \mathbf{u})(1 + \mathbf{x})\,. \tag{83}$$

So, now everything hinges on how we can get rid of the $\mathbf{e}_0$'s in the first term on the RHS. With the understanding that the vectors $\mathbf{a}$ and $\mathbf{u}$ are orthogonal to $\mathbf{e}_0$, and that $\mathbf{e}^2 = 1$, we get

$$\begin{aligned} \mathbf{e}_0(\mathbf{u} + \mathbf{a} \wedge \mathbf{u})\mathbf{e}_0 &= \mathbf{e}_0\mathbf{u}\mathbf{e}_0 + \mathbf{e}_0\mathbf{a} \wedge \mathbf{u}\mathbf{e}_0 \\ &= (2\mathbf{e}_0 \cdot \mathbf{u} - \mathbf{u}\mathbf{e}_0)\mathbf{e}_0 + \langle\, \mathbf{e}_0\mathbf{a} \wedge \mathbf{u}\mathbf{e}_0\,\rangle_2 \\ &= -\mathbf{u} + \mathbf{e}_0 \cdot (\mathbf{a} \wedge \mathbf{u} \wedge \mathbf{e}_0) \\ &= -\mathbf{u} + \mathbf{a} \wedge \mathbf{u}\,. \end{aligned} \tag{84}$$

On substituting this result into (83), we get

$$\begin{aligned} \Omega &= (1 + \mathbf{x})(-\mathbf{u} + \mathbf{a} \wedge \mathbf{u}) + (\mathbf{u} + \mathbf{a} \wedge \mathbf{u})(1 + \mathbf{x}) \\ &= -\mathbf{u} + \mathbf{a} \wedge \mathbf{u} - \mathbf{x}\mathbf{u} + \mathbf{x}\mathbf{a} \wedge \mathbf{u} + \mathbf{u} + \mathbf{u}\mathbf{x} \\ &\qquad + \mathbf{a} \wedge \mathbf{u} + \mathbf{a} \wedge \mathbf{u}\mathbf{x} \\ &= 2\mathbf{a} \wedge \mathbf{u} + (\mathbf{u}\mathbf{x} - \mathbf{x}\mathbf{u}) + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} \\ &= 2\mathbf{a} \wedge \mathbf{u} + -2\mathbf{x} \wedge \mathbf{u} + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} \\ &= 2(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,, \end{aligned} \tag{85}$$

where, on going between steps 2 and 3, we did a lot of cancellation, using, in particular, (76). Hence, we have (81) becoming

$$2x \wedge B\mathbf{e}_0/x_0a_0b_0 = 2(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + 2\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,. \tag{86}$$

From this we get

$$x \wedge B = x_0a_0b_0[\,(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0\,. \tag{87}$$

Using that $B = a \wedge b$, we have that

$$x \wedge a \wedge b = x_0 a_0 b_0 [\,(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}\,]\mathbf{e}_0\,. \tag{88}$$

For $x \wedge a \wedge b$ to vanish, we need

$$(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} = 0\,. \tag{89}$$

Now we have arrived at a fun part of the paper: the part that proves the invariance of the **cross ratio**. So, we start with three distinct points a, b, c on a given line in $\mathcal{P}_2$. (We deduce that the wedge product of any two of them is a nonzero scalar multiple of the wedge product of any other two of them.)

So, if we can show that

$$b_0 a \wedge c(\mathbf{b} - \mathbf{c}) = a_0 (b \wedge c)(\mathbf{a} - \mathbf{c})\,, \tag{90}$$

then we can write

$$\frac{a \wedge c}{b \wedge c} = \frac{a_0(\mathbf{a} - \mathbf{c})}{b_0(\mathbf{b} - \mathbf{c})} = \frac{a_0}{b_0}\frac{\mathbf{a} - \mathbf{c}}{\mathbf{b} - \mathbf{c}}\,, \tag{91}$$

which is Eq. (39) in the preprint paper.

In preparation, we need a couple results first. For one, $\mathbf{b} - \mathbf{c}$ is related to $\mathbf{a} - \mathbf{c}$ by a factor of a nonzero scalar multiple,[8] say α, or

$$\mathbf{b} - \mathbf{c} = \alpha(\mathbf{a} - \mathbf{c})\,. \tag{92}$$

Now, on wedging this last result by $\mathbf{c}$ on the left, we get the next result

$$\mathbf{c} \wedge \mathbf{b} = \alpha \mathbf{c} \wedge \mathbf{a}\,. \tag{93}$$

We also need the following lemma. Starting with

$$a \wedge c = a_0 c_0 (\mathbf{a} - \mathbf{c} + \mathbf{c} \wedge \mathbf{a})\,, \tag{94}$$

we get that

$$\begin{aligned} \alpha(a \wedge c) &= a_0 c_0 (\alpha(\mathbf{a} - \mathbf{c}) + \alpha \mathbf{c} \wedge \mathbf{a}) \\ &= a_0 c_0 ((\mathbf{b} - \mathbf{c}) + \mathbf{c} \wedge \mathbf{b}) \\ &= \frac{a_0}{b_0} b_0 c_0 ((\mathbf{b} - \mathbf{c}) + \mathbf{c} \wedge \mathbf{b}) \\ &= \frac{a_0}{b_0} b \wedge c\,. \end{aligned} \tag{95}$$

So, let's start on the LHS of (90) and proceed to the RHS.

$$\begin{aligned} b_0 a \wedge c\,(\mathbf{b} - \mathbf{c}) &= a \wedge c\, b_0 \alpha(\mathbf{a} - \mathbf{c}) \\ &= \alpha(a \wedge c)\, b_0 (\mathbf{a} - \mathbf{c}) \\ &= \frac{a_0}{b_0} b \wedge c\, b_0 (\mathbf{a} - \mathbf{c}) \quad \text{(using (95))} \\ &= b \wedge c\, a_0 (\mathbf{a} - \mathbf{c}) \\ &= a_0 (b \wedge c)(\mathbf{a} - \mathbf{c})\,. \end{aligned} \tag{96}$$

[8]This is because points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ lie on the same line.

However, Eq. (91) has the scalars a_0 and b_0, thus this relation, being based on only three points, is not the invariant relationship we seek. To find it, let's substitute d for c (where d is yet another distinct point on the same line) in (91) to get

$$\frac{a \wedge d}{b \wedge d} = \frac{a_0(\mathbf{a} - \mathbf{d})}{b_0(\mathbf{b} - \mathbf{d})} = \frac{a_0}{b_0} \frac{\mathbf{a} - \mathbf{d}}{\mathbf{b} - \mathbf{d}}\,, \tag{97}$$

If we now divide (91) by (97) we get

$$\frac{a \wedge c}{b \wedge c} \frac{b \wedge d}{a \wedge d} = \frac{\mathbf{a} - \mathbf{c}}{\mathbf{b} - \mathbf{c}} \frac{\mathbf{b} - \mathbf{d}}{\mathbf{a} - \mathbf{d}}\,, \tag{98}$$

which is Eq. (40) of the preprint paper and is also the invariant cross ratio, based on four distinct points.

8 Conclusion

There is more to this article, but this is as far as I intend to go into it at this time. The reason is because there are other Hestenes articles and presentations by Chris Doran and others that are more extensive and clearer.

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