Theorem: The Angle Bisectors of a Triangle Are Concurrent

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Abstract

This paper uses Geometric Algebra to prove the theorem that the angle bisectors of a triangle are concurrent.

1 Introduction

First, a definition: Three or more lines in a plane are said to be *concurrent* if they all meet at a single point.

This problem has two sister problems, namely: 1) The altitudes of a triangle are concurrent. 2) The medians of a triangle are concurrent. There is a common technique to solving all these problems: Any two of the line segments are bound to intersect in a point, and the trick is to show that the third line segment passes through that point.[1]

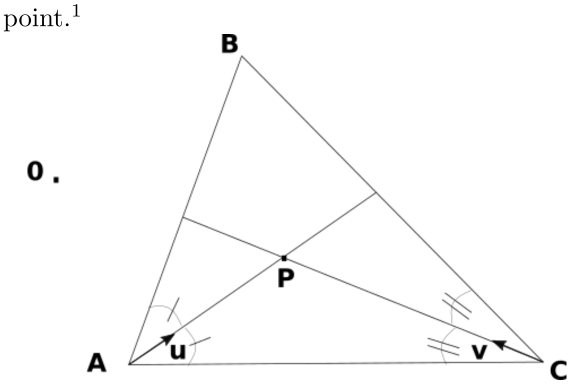

Figure 1. The angle bisectors at vertices **A** and **C** intersect at point **P**. The indicated vectors **u** and **v** are collinear with the indicated lines of angle bisection. I included the explicit representation of the vector spaces's origin **O**. It's position is almost arbitrary.

[1] I'm **not** suggesting that the general procedure outlined here is the only way to solve these problems.

2 Step One

We know that the angle bisectors from vertices **A** and **C** will meet in some point. I called it **P**. Our task now is to show that the angle bisector from vertex **B** will also go through point **P**.

We begin this task by representing the vectors **u** and **v** as functions of the three vertices. The way to make **u** a vector that bisects the angle at **A** is to form it out of equal amounts of vectors along directions **B** – **A** and **C** – **A**, and this is easy to do:

$$\mathbf{u} \equiv \frac{\mathbf{B}-\mathbf{A}}{|\,\mathbf{B}-\mathbf{A}\,|} + \frac{\mathbf{C}-\mathbf{A}}{|\,\mathbf{C}-\mathbf{A}\,|}\,, \tag{1}$$

where each term on the RHS is a unit vector. As unit vectors, they have the same magnitudes, thus they balance **u** between the two sides. Similarly,

$$\mathbf{v} \equiv \frac{\mathbf{A}-\mathbf{C}}{|\,\mathbf{A}-\mathbf{C}\,|} + \frac{\mathbf{B}-\mathbf{C}}{|\,\mathbf{B}-\mathbf{C}\,|}\,. \tag{2}$$

Now, neither **u** nor **v** are themselves unit vectors, but that won't matter.

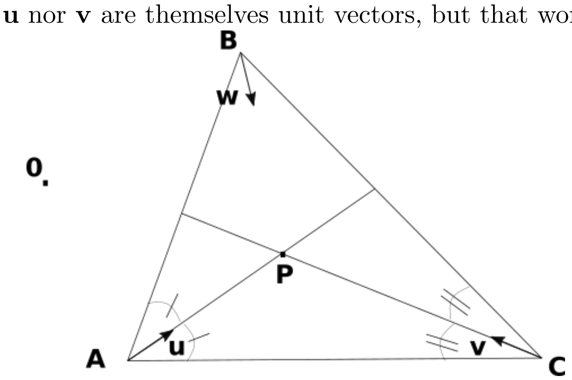

Figure 2. The angle bisector at vertex **B** is now added. Is it along the line segment from **B** and including **P**?

On writing **w** similarly to how we wrote the previous two bisecting vectors, we get

$$\mathbf{w} \equiv \frac{\mathbf{A}-\mathbf{B}}{|\,\mathbf{A}-\mathbf{B}\,|} + \frac{\mathbf{C}-\mathbf{B}}{|\,\mathbf{C}-\mathbf{B}\,|}\,. \tag{3}$$

Now, upon adding Equations (1), (2), and (3), we get

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = 0\,. \tag{4}$$

It's quite tempting to say that because of the obvious symmetry of this last equation, then **w** must be on the line joining **B** to **P**, but for a more rigorous proof, let's continue.

3 Step Two

As a reminder, we need to show that the vector $\mathbf{P} - \mathbf{B}$ is parallel to vector $\mathbf{w}$, or,

$$\mathbf{P} - \mathbf{B} = \alpha \mathbf{w}\,, \tag{5}$$

where α is some real number. Let's begin by writing down an equation for $\mathbf{P}$:

$$\mathbf{P} = \mathbf{A} + \lambda \mathbf{u} = \mathbf{C} + \lambda' \mathbf{v}\,, \tag{6}$$

where λ and λ' are real numbers. In other words, we can get to $\mathbf{P}$ either by starting at the origin and going to $\mathbf{A}$ and then traversing $\lambda\mathbf{u}$ to arrive at $\mathbf{P}$, or else, by starting at the origin and going to $\mathbf{C}$ and then traversing $\lambda'\mathbf{v}$ to arrive at to $\mathbf{P}$.

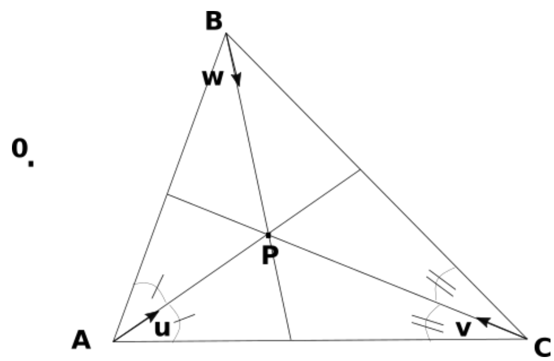

Figure 3. The angle bisector at vertex **B** is now added to the figure. Is it along the line segment from **B** and including **P**?

Eq. (6) can be rearranged to

$$\lambda \mathbf{u} - \lambda' \mathbf{v} = \mathbf{C} - \mathbf{A}\,. \tag{7}$$

On wedging this through by $\mathbf{v}$, we get

$$\lambda \mathbf{u} \wedge \mathbf{v} = (\mathbf{C} - \mathbf{A}) \wedge \mathbf{v}\,, \tag{8}$$

which will be used later.[2]

4 Step Three: Sleight of Hand

Question: When is cheating not really cheating? Answer: when it's done by **virtual emplacement, VE**. This is the technique of adding in and subtracting out the same quantity or by multiplying by some amount that gets divided out by the same amount. (More generally, the application of some invertible function and then its inverse.)

[2] We haven't been asked to solve for λ, so we need a simple way to get rid of it.

We'll be using the following technique at a crucial moment later on. Let's look at a generic case. Consider the vector equation:

$$\mathbf{H} = (\mathbf{A} - \mathbf{B}) \wedge (\mathbf{C} - \mathbf{B})\,. \tag{9}$$

Suppose, however, that instead of $\mathbf{H}$ being the wedge of $(\mathbf{A} - \mathbf{B})$ and $(\mathbf{C} - \mathbf{B})$, we wanted it to be of the form

$$\mathbf{H}' = (\mathbf{A} - \mathbf{C}) \wedge (\mathbf{C} - \mathbf{B})\,. \tag{10}$$

It is easy to migrate from $\mathbf{H}$ to $\mathbf{H}'$ as follows. Let's go back to Eq. (9):

$$\begin{aligned}
\mathbf{H} &= [\mathbf{A} - \mathbf{B}\,] \wedge (\mathbf{C} - \mathbf{B}) \\
&= [\,(\mathbf{A} - \mathbf{C}) + (\mathbf{C} - \mathbf{B})\,] \wedge (\mathbf{C} - \mathbf{B}) \quad \text{(by VE)} \\
&= (\mathbf{A} - \mathbf{C}) \wedge (\mathbf{C} - \mathbf{B}) + (\mathbf{C} - \mathbf{B}) \wedge (\mathbf{C} - \mathbf{B}) \\
&= (\mathbf{A} - \mathbf{C}) \wedge (\mathbf{C} - \mathbf{B}) \\
&= \mathbf{H}'\,.
\end{aligned} \tag{11}$$

In other words,

$$(\mathbf{A} - \mathbf{B}) \wedge (\mathbf{C} - \mathbf{B}) = (\mathbf{A} - \mathbf{C}) \wedge (\mathbf{C} - \mathbf{B})\,. \tag{12}$$

5 Step Four: The proof proper

Combining Eqs. (5) and (6), we get

$$\mathbf{P} - \mathbf{B} = \mathbf{A} - \mathbf{B} - \lambda\mathbf{u} = \alpha\mathbf{w}\,, \tag{13}$$

By wedging through by $\mathbf{w}$, we get the equivalent equation to establish:

$$(\mathbf{A} - \mathbf{B} - \lambda\mathbf{u}) \wedge \mathbf{w} = (\mathbf{A} - \mathbf{B}) \wedge \mathbf{w} - \lambda\mathbf{u} \wedge \mathbf{v} = 0\,, \tag{14}$$

where we used Eq. (4).

$$\begin{aligned}
(\mathbf{P} - \mathbf{B}) \wedge \mathbf{w} &= (\mathbf{A} - \mathbf{B}) \wedge \mathbf{w} - (\mathbf{C} - \mathbf{A}) \wedge \mathbf{v} \quad \text{(from (8))} \\
&= (\mathbf{A} - \mathbf{B}) \wedge \left[\frac{\mathbf{A} - \mathbf{B}}{|\,\mathbf{A} - \mathbf{B}\,|} + \frac{\mathbf{C} - \mathbf{B}}{|\,\mathbf{C} - \mathbf{B}\,|}\right] \\
&\qquad - (\mathbf{C} - \mathbf{A}) \wedge \left[\frac{\mathbf{A} - \mathbf{C}}{|\,\mathbf{A} - \mathbf{C}\,|} + \frac{\mathbf{B} - \mathbf{C}}{|\,\mathbf{B} - \mathbf{C}\,|}\right] \\
&= (\mathbf{A} - \mathbf{B}) \wedge \frac{\mathbf{C} - \mathbf{B}}{|\,\mathbf{C} - \mathbf{B}\,|} - (\mathbf{C} - \mathbf{A}) \wedge \frac{\mathbf{B} - \mathbf{C}}{|\,\mathbf{B} - \mathbf{C}\,|} \\
&= (\mathbf{A} - \mathbf{C}) \wedge \frac{\mathbf{C} - \mathbf{B}}{|\,\mathbf{C} - \mathbf{B}\,|} - (\mathbf{C} - \mathbf{A}) \wedge \frac{\mathbf{B} - \mathbf{C}}{|\,\mathbf{B} - \mathbf{C}\,|} \qquad \text{(by VE)} \\
&= 0\,.
\end{aligned} \tag{15}$$

Done.

6 A couple extra problems for the fun of it

Solve for λ in (6) and α in (5).

I chose to solve for λ in (8)

$$\lambda \mathbf{u} \wedge \mathbf{v} = (\mathbf{C} - \mathbf{A}) \wedge \mathbf{v}\,. \tag{16}$$

I won't present the full proof here, but I'll give hints. Step 1) Use Eqs. (1) and (2), and substitute into this last equation. Along the way, make a choice of how to represent both $\mathbf{u} \wedge \mathbf{v}$ and $(\mathbf{C} - \mathbf{A}) \wedge \mathbf{v}$ as the same bivector. I chose the bivector $(\mathbf{B} - \mathbf{A}) \wedge (\mathbf{A} - \mathbf{C})$, eventually arriving at

$$\begin{aligned} \lambda(\mathbf{B} - \mathbf{A}) \wedge (\mathbf{A} - \mathbf{C}) &\left[\frac{1}{|\,\mathbf{B} - \mathbf{A}\,|\,|\,\mathbf{A} - \mathbf{C}\,|} + \frac{1}{|\,\mathbf{B} - \mathbf{A}\,|\,|\,\mathbf{B} - \mathbf{C}\,|} + \frac{1}{|\,\mathbf{A} - \mathbf{C}\,|\,|\,\mathbf{B} - \mathbf{C}\,|} \right] \\ &= \frac{(\mathbf{B} - \mathbf{A}) \wedge (\mathbf{A} - \mathbf{C})}{|\,\mathbf{B} - \mathbf{C}\,|}\,, \end{aligned} \tag{17}$$

in which we used the virtual emplacement trick.

Next, we multiply (17) through by $|\,\mathbf{B} - \mathbf{A}\,|\,|\,\mathbf{A} - \mathbf{C}\,|\,|\,\mathbf{B} - \mathbf{C}\,|$ and drop the bivector factors,[3] to get

$$\lambda[\,|\,\mathbf{B} - \mathbf{C}\,| + |\,\mathbf{A} - \mathbf{C}\,| + |\,\mathbf{B} - \mathbf{A}\,|\,] = |\,\mathbf{B} - \mathbf{A}\,|\,|\,\mathbf{A} - \mathbf{C}\,|\,, \tag{18}$$

from which we get

$$\lambda = \frac{|\,\mathbf{B} - \mathbf{A}\,|\,|\,\mathbf{A} - \mathbf{C}\,|}{|\,\mathbf{B} - \mathbf{C}\,| + |\,\mathbf{A} - \mathbf{C}\,| + |\,\mathbf{B} - \mathbf{A}\,|}\,, \tag{19}$$

As for α, we can combine to Eqs. (5) and (6), and just solve the result for α:

$$\alpha \mathbf{w} = \mathbf{A} - \mathbf{B} + \lambda \mathbf{u}\,, \tag{20}$$

yielding[4]

$$\alpha = (\mathbf{A} - \mathbf{B} + \lambda \mathbf{u}) \cdot \mathbf{w}^{-1}\,. \tag{21}$$

Alternatively, we could wedge (20) through by $\mathbf{u}$ on the right, to get

$$\alpha \mathbf{w} \wedge \mathbf{u} = (\mathbf{A} - \mathbf{B}) \wedge \mathbf{u}\,, \tag{22}$$

and just solve for α similarly to how we solved for λ, to get

$$\alpha = \frac{|\,\mathbf{B} - \mathbf{C}\,|\,|\,\mathbf{A} - \mathbf{B}\,|}{|\,\mathbf{B} - \mathbf{C}\,| + |\,\mathbf{A} - \mathbf{C}\,| + |\,\mathbf{B} - \mathbf{A}\,|}\,. \tag{23}$$

Done.

[3] All nonzero bivectors in this algebra have inverses.
[4] Expanding the RHS of this is difficult.