Notes on *Local Observables in the Dirac Theory*

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Abstract

This paper contains my personal notes on the paper *Local Observables in the Dirac Theory* by David Hestenes.[1] My comments are meant 1) to fill-in some of the steps in the mathematical derivations, and 2) to report on a few mistakes that remain in the "preprint" version of the paper. As a word of warning, this paper will make no attempt to teach the full fundamentals of geometric/Clifford algebra, though it will spend some time enhancing the discussion on it presented in the paper. Nevertheless, this paper emphasizes the mathematics rather than the physics.

Introduction

The paper referenced is available on line as a "preprint" at

`http://geocalc.clas.asu.edu/html/GAinQM.html` .

However, the preprint has some errors in it and I shall endevour to point them out as I find them. However, the published article has a few errors as well, and I'll point them out too, as I find them.

I want to point out that I consulted the following paper (the first three pages) by Stephen Gull when I sought to correct the errors in the "preprint" version of the paper:

`https://www.mrao.cam.ac.uk/~steve/MONOPOLE.pdf` .

I assume that the reader has a pretty good grasp on geometric algebra, as this is necessary to follow the presentation. There are many books that can be purchased to learn this algebra, and many on-line articles — some free, some not — that can be acquired.

[1] D. Hestenes, "Local Observables in the Dirac Theory," *J. Math. Phys.* Vol. **14**, July (1973). p.893–905

1 The Wave Function

We introduce our formulation of the wavefunction as follows (Eq. (1.1)):

$$\psi = \rho^{1/2} e^{\frac{1}{2} i\beta} R \,, \tag{1}$$

where ρ and β are scalars of the theory, which will be explained later. R is an even multivector.

We can think of Eq. (1) as a canonical factorization of the wavefunction in analogy to how one factors matrices into some canonical factorization.[2] We'll properly define i soon.

We think of R as a Lorentz rotation operator with constraint

$$R\widetilde{R} = 1 \,, \tag{2}$$

where the ˜ operator reverses the ordering of all geometric products. Scalars and vectors are, of course, invariant under the reversion operation. We get that

$$\psi\widetilde{\psi} = \rho e^{i\beta} \,, \tag{3}$$

where ρ is the proper probabillity density.

Next, we define a mutually orthogonal set of vectors γ_μ ($\mu = 0, 1, 2, 3$) to form a frame, with $\gamma_0^2 = 1$ and the rest square to -1.

The following are basic results that we'll use throughout the rest of the paper:

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 \,. \tag{4}$$

$$i^\sim = i \,. \tag{5}$$

The proof of this is that it requires an even number of transpositions to bring

$$i^\sim = \gamma_3\gamma_2\gamma_1\gamma_0 \tag{6}$$

back to the original form $\gamma_0\gamma_1\gamma_2\gamma_3 = i$.

As a consequence

$$(e^{i\beta})^\sim = e^{i\beta} \,. \tag{7}$$

For all γ_μ ($\mu = 0, 1, 2, 3$):

$$\gamma_\mu i = -i\gamma_\mu \,, \tag{8}$$

and as a consequence,

$$\gamma_\mu e^{i\beta} = \gamma_\mu(\cos\beta + i\sin\beta) = (\cos\beta - i\sin\beta)\gamma_\mu = e^{-i\beta}\gamma_\mu \,. \tag{9}$$

The function of the spinor R is to rotate this frame according to the rule

$$e_\alpha = R\gamma_\alpha\widetilde{R} \qquad (\alpha = 0, 1, 2, 3) \,, \tag{10}$$

[2] See Appendix A for a correlation of the conventional matrix version of the Dirac theory to the geometric algebra version.

which is Eq. (1.2). It's pretty easy to see that if B is any bivector, then $Bi = iB$. And, as a consequent, $e^{i\beta}B = Be^{i\beta}$. It's likewise easy to see that for any even multivector $M = \langle M \rangle_+$, $Mi = iM$ and $e^{i\beta}M = Me^{i\beta}$, which is why $e^{i\beta}$ commutes with R, and for that matter with $\widetilde{R}$.

By convention, the vectors $e_0 = v$ and e_3 are granted special physical meaning in the theory. The first is the local particle velocity in spacetime and the second is the local spin direction. My question now is, How much does this conventionalization restrict the choice of how to generalize the Dirac equation (24) to the first generalization of it here in Eq. (25)?

We find the probability current as

$$\psi\gamma_0\widetilde{\psi} = \rho R\gamma_0\widetilde{R} = \rho v\,, \tag{11}$$

which is Eq. (1.3). Let's prove this.

$$\begin{aligned}\psi\gamma_0\widetilde{\psi} &= (\rho^{1/2}e^{\frac{1}{2}i\beta}R)\gamma_0(\rho^{1/2}e^{\frac{1}{2}i\beta}R)^{\sim}\\ &= (\rho^{1/2}e^{\frac{1}{2}i\beta}R)\gamma_0(\widetilde{R}(e^{\frac{1}{2}i\beta})^{\sim}\rho^{1/2})\\ &= \rho(e^{\frac{1}{2}i\beta}R)\gamma_0(\widetilde{R}e^{\frac{1}{2}i\beta}) = \rho e^{\frac{1}{2}i\beta}R\gamma_0 e^{\frac{1}{2}i\beta}\widetilde{R}\\ &= \rho e^{\frac{1}{2}i\beta}e^{-\frac{1}{2}i\beta}R\gamma_0\widetilde{R}\\ &= \rho R\gamma_0\widetilde{R} = \rho v\,.\end{aligned} \tag{12}$$

For the local conservation of probability we have Eq. (1.4):

$$\Box\cdot(\rho v) = 0\,. \tag{13}$$

The proper mass density is $m\rho$. The local spin vector is Eq. (1.5):

$$s = \frac{\hbar}{2}e_3\,, \tag{14}$$

which converts the 'physical spin direction' e_3 to an angular momentum vector s. We define the bivector version of the spin as

$$S \equiv isv = \frac{\hbar}{2}ie_3e_0 = \frac{\hbar}{2}e_2e_1 = \frac{\hbar}{2}R\gamma_2\gamma_1\widetilde{R}\,, \tag{15}$$

which is Eq. (1.6).

Although the frame of γ_μ's is arbitrary, the instantaneous comoving frame of γ_μ's is intrinsic to the electron itself, and the rotor R connects them.

Equation (1.7) in the article is given as

$$R = (\widetilde{A}A)^{-1/2}A\,, \tag{16}$$

for some field A. We show that $R\widetilde{R} = 1$:

$$\begin{aligned}R\widetilde{R} &= (\widetilde{A}A)^{-1/2}A[(\widetilde{A}A)^{-1/2}A]^{\sim}\\ &= (\widetilde{A}A)^{-1/2}A\widetilde{A}(\widetilde{A}A)^{-1/2}\\ &= (\widetilde{A}A)^{-1/2}A\widetilde{A}(\widetilde{A}A)^{-1/2}\\ &= (\widetilde{A}A)^{-1}A\widetilde{A}\\ &= 1\,.\end{aligned} \tag{17}$$

At this point I want to perform a calculation. But I need some preliminary results first. Let's begin with this: Anytime we take the scalar part of a geometric product of multivectors, we can cyclicly permute them as follows:

$$\langle ABCD\cdots Z\rangle = \langle BCD\cdots ZA\rangle = \langle CD\cdots ZAB\rangle = \text{etc.}\,. \tag{18}$$

And, of course, we can also cyclically permute going the opposite direction.

So, I wish to prove that $s\cdot v = 0$. This makes sense, since v is a timelike vector in a given frame and s is a spacelike vector in the same frame.

Also, since $\psi\widetilde{\psi} = \rho e^{i\beta}$, then

$$\widetilde{\psi}(\psi\widetilde{\psi})\psi = \widetilde{\psi}(\rho e^{i\beta})\psi = \widetilde{\psi}\psi(\rho e^{i\beta})\,, \tag{19}$$

or

$$(\widetilde{\psi}\psi)^2 = \widetilde{\psi}\psi(\rho e^{i\beta})\,, \tag{20}$$

Barring the case that $\widetilde{\psi}\psi = 0$, then

$$\widetilde{\psi}\psi = \rho e^{i\beta}\,. \tag{21}$$

So,

$$\begin{aligned} s\cdot v &= (\frac{\hbar}{2}\psi\gamma_3\widetilde{\psi})\cdot(\psi\gamma_0\widetilde{\psi}) \\ &= \frac{\hbar}{2}\langle\,\psi\gamma_3\widetilde{\psi}\psi\gamma_0\widetilde{\psi}\,\rangle \\ &= \frac{\hbar}{2}\langle\,\psi\gamma_3(\rho e^{i\beta})\gamma_0\widetilde{\psi}\,\rangle \\ &= \frac{\hbar}{2}\langle\,\widetilde{\psi}\psi(\rho e^{-i\beta})\gamma_3\gamma_0\,\rangle \\ &= \frac{\hbar\rho^2}{2}\langle\,e^{i\beta}(e^{-i\beta})\gamma_3\wedge\gamma_0\,\rangle \\ &= \frac{\rho^2\hbar}{2}\langle\,\gamma_3\wedge\gamma_0\,\rangle \\ &= 0\,, \end{aligned} \tag{22}$$

since the contents inside the selector is pure bivector. (Remember that the γ_μ's are an orthormal set, given that $\gamma^\mu\cdot\gamma_\nu = \delta^\mu_\nu$.)

One more result. Let's show that $v\cdot S = 0$, using that $v^2 = 1$.

$$\begin{aligned} v\cdot S &= v\cdot(isv) = \langle\,visv\,\rangle \\ &= \langle\,vvis\,\rangle = \langle\,is\,\rangle \\ &= 0\,. \end{aligned} \tag{23}$$

2 Energy-Momentum Tensor

According to Dirac, the total energy of an electron in a stationary state is given as

$$(-1)^{1/2}\hbar\partial_t\Psi = E\Psi\,, \tag{24}$$

which is Eq. (2.1). Of course, in this last equation, Ψ is a column spinor. But in the geometric algebra of spacetime, we can rewrite this as Eq. (2.2):

$$\partial_t\psi\gamma_2\gamma_1\hbar = E\psi\,, \tag{25}$$

where $\gamma_2\gamma_1$ is a bivector that defines a plane of rotation, can be re-expressed as

$$\gamma_2\gamma_1 = i\gamma_3\gamma_0 = i\sigma_3\,. \tag{26}$$

We 'guess' at a proper generalization of (24) as Eq. (2.3):

$$\begin{aligned}T_{\mu\nu} &= \langle\,\gamma_0\widetilde{\psi}\gamma_\mu(\partial_\nu\psi\gamma_2\gamma_1\hbar - eA_\nu\psi)\,\rangle \\ &= \hbar\langle\,\gamma_\mu(\partial_\nu\psi)\gamma_2\gamma_1\gamma_0\widetilde{\psi}\,\rangle - eA_\nu\langle\,\gamma_0\widetilde{\psi}\gamma_\mu\psi\,\rangle \\ &= \hbar\langle\,\gamma_\mu(\partial_\nu\psi)i\gamma_3\widetilde{\psi}\,\rangle - eA_\nu\langle\,(\psi\gamma_0\widetilde{\psi})\gamma_\mu\,\rangle \\ &= \hbar\langle\,\gamma_\mu(\partial_\nu\psi)i\gamma_3\widetilde{\psi}\,\rangle - e\rho v_\mu A_\nu\,,\end{aligned} \tag{27}$$

where we have corrected a typo in the first line involving the gamma next to the $\hbar$. So, let's investigate how this demonstration works. In the third line we used (11) and the fact that

$$\langle\,v\gamma_\mu\,\rangle = v\cdot\gamma_\mu = v_\mu\,. \tag{28}$$

The average energy in inertial system γ_0 is

$$\langle\,E\,\rangle = \int d^3x\,(T_{00} + e\rho v_0A_0)\,. \tag{29}$$

If we rewrite (27) as

$$T_{\mu\nu} + e\langle\,\gamma_0\widetilde{\psi}\gamma_\mu A_\nu\,\rangle = \hbar\langle\,\gamma_\mu(\partial_\nu\psi)\gamma_2\gamma_1\gamma_0\widetilde{\psi}\,\rangle\,, \tag{30}$$

and then set $\mu = \nu = 0$, we get [Note: $\partial_0 = \partial_t$]

$$T_{00} + e\langle\,\gamma_0\widetilde{\psi}\gamma_0A_0\,\rangle = \hbar\langle\,\gamma_0(\partial_0\psi)\gamma_2\gamma_1\gamma_0\widetilde{\psi}\,\rangle\,, \tag{31}$$

or

$$T_{00} + e\rho v_0A_0 = \langle\,\gamma_0(\partial_0\psi)\gamma_2\gamma_1\gamma_0\widetilde{\psi}\hbar\,\rangle = \langle\,\gamma_0\widetilde{\psi}\gamma_0(\partial_0\psi)\gamma_2\gamma_1\hbar\,\rangle\,. \tag{32}$$

Therefore, (29) becomes

$$\begin{aligned}\langle\,E\,\rangle = \int d^3x\,(T_{00} + e\rho v_0A_0) &= \int d^3x\,\langle\,\gamma_0\widetilde{\psi}\gamma_0(\partial_0\psi)\gamma_2\gamma_1\hbar\,\rangle \\ &= E\int d^3x\,\langle\,\gamma_0\widetilde{\psi}\gamma_0\psi\,\rangle\,,\end{aligned} \tag{33}$$

since

$$\partial_0 \psi \gamma_2 \gamma_1 \hbar = E\psi \,. \tag{34}$$

But

$$\langle \gamma_0 \widetilde{\psi} \gamma_0 \psi \rangle = \langle \psi \gamma_0 \widetilde{\psi} \gamma_0 \rangle = \langle \rho v \gamma_0 \rangle = \rho v_0 \equiv \rho_0 \,, \tag{35}$$

which is Eq. (2.6). Therefore, (33) becomes

$$\langle E \rangle = E \int d^3x \, \rho_0 = E \,, \tag{36}$$

which is Eq. (2.5).

We are now to review Tetrode's version of the electron's energy-momentum tensor in geometric algebra. See Appendix A for the matrix version.

$T(n)$ is a linear tensor that denotes the energy-momentum flux through a hypersurface of nornal n. Hence,

$$T(n) = n^\mu T(\gamma_\mu) = n^\mu T_\mu \,, \tag{37}$$

which is Eq. (2.7). Obviously, T_μ is a vector quantity. To get the 16 components of T, we have

$$T_{\mu\nu} = T_\mu \cdot \gamma_\nu \quad \text{and} \quad T_\mu = T_{\mu\nu} \gamma^\nu \,, \tag{38}$$

which is Eq. (2.8a,b). The proper energy-momentum density is

$$\rho p = T(v) = v^\mu T_\mu \,, \tag{39}$$

which is Eq. (2.9). We can write T_μ as (2.10):

$$T_\mu = \rho v_\mu p + N_\mu \,, \tag{40}$$

where N_μ is normal to the streamlines, and, of course, $v^\mu v_\mu = 1$. The N_μ are as yet unconstrained degrees of freedom, except that we require that $v^\mu N_\mu = 0$ so that we can recoup (39) from (40). We'll return to this tensor later.

We define the "transposed" tensor of $T_{\mu\nu}$ (Eq. (2.11)) as

$$\begin{aligned} \bar{T}_\mu = \gamma^\nu T_{\mu\nu} &= \gamma^\nu [\, \hbar \gamma_\nu \cdot \langle \partial_\mu \psi i \gamma_3 \widetilde{\psi} \rangle_1 - e\rho v_\nu A_\mu \,] \\ &= \hbar \langle \partial_\mu \psi i \gamma_3 \widetilde{\psi} \rangle_1 - e\rho v A_\mu \,, \end{aligned} \tag{41}$$

where the v in the right term on the bottom line does not have a subscript, and thus corrects a typo.

However, this does not yield the correct result using the value for $T_{\mu\nu}$ as given in (27).

To get Eq. (2.3) of the paper, I used

$$\begin{aligned} T_{\mu\nu} &= \langle \gamma_0 \widetilde{\psi} \gamma_\mu (\partial_\nu \psi) \gamma_2 \gamma_1 \hbar - e A_\nu \psi \rangle \\ &= \langle \gamma_0 \widetilde{\psi} \gamma_\mu (\partial_\nu \psi) \gamma_2 \gamma_1 \hbar \rangle - \langle e A_\nu \gamma_0 \widetilde{\psi} \gamma_\mu \psi \rangle \\ &= \hbar \langle \widetilde{\psi} \gamma_\mu (\partial_\nu \psi) \gamma_2 \gamma_1 \gamma_0 \rangle - \langle e A_\nu \psi \gamma_0 \widetilde{\psi} \gamma_\mu \rangle \\ &= \hbar \langle \widetilde{\psi} \gamma_\mu (\partial_\nu \psi) i \gamma_3 \rangle - e\rho \langle A_\nu v \gamma_\mu \rangle \\ &= \hbar \langle \gamma_\mu (\partial_\nu \psi) i \gamma_3 \widetilde{\psi} \rangle - e\rho A_\nu \langle v \gamma_\mu \rangle \,. \end{aligned} \tag{42}$$

So, we have the special case

$$T_{00} = \hbar\langle\, \gamma_0(\partial_0\psi)i\gamma_3\widetilde{\psi}\,\rangle - e\rho A_0\langle\, v\gamma_0\,\rangle\,, \tag{43}$$

Or,

$$T_{00} + eA_0\rho v_0 = \hbar\langle\, \gamma_0(\partial_0\psi)i\gamma_3\widetilde{\psi}\,\rangle\,, \tag{44}$$

where $v_0 = \langle\, v\gamma_0\,\rangle$.

Then,

$$\begin{aligned}\bar{T}_\mu \equiv \gamma^\nu T_{\mu\nu} &= \gamma^\nu[\,\hbar\langle\, \gamma_\nu(\partial_\mu\psi)i\gamma_3\widetilde{\psi}\,\rangle - e\rho v_\nu A_\mu\,]\\ &= \hbar\gamma^\nu\langle\, \gamma_\nu(\partial_\mu\psi)i\gamma_3\widetilde{\psi}\,\rangle - e\rho v A_\mu\\ &= \hbar\gamma^\nu\gamma_\nu\cdot\langle\,(\partial_\mu\psi)i\gamma_3\widetilde{\psi}\,\rangle_1 - e\rho v A_\mu\\ &= \hbar\langle\,(\partial_\mu\psi)i\gamma_3\widetilde{\psi}\,\rangle_1 - e\rho v A_\mu\end{aligned} \tag{45}$$

For our next result, we employ a simple identity. Let A be vector valued. Then,

$$2A = A + \widetilde{A}\,, \tag{46}$$

since $\widetilde{A} = A$. Note that $\widetilde{i} = i$ and $i\gamma_\mu = -\gamma_\mu i$, and for any multivector M, $\widetilde{M}^{\sim} = M$:

$$\begin{aligned}2\langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 &= \langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 + \langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1^{\sim}\\ &= \langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 + \langle\, \psi\gamma_3\widetilde{i}\partial_\mu\widetilde{\psi}\,\rangle_1\\ &= \langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 + \langle\, \psi\gamma_3 i\partial_\mu\widetilde{\psi}\,\rangle_1\\ &= \langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 - \langle\, \psi i\gamma_3\partial_\mu\widetilde{\psi}\,\rangle_1\,.\end{aligned} \tag{47}$$

Therefore, we've arrived at Eq. (2.12):

$$\hbar\langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 = \frac{\hbar}{2}[\,\langle\, \partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 - \langle\, \psi i\gamma_3\partial_\mu\widetilde{\psi}\,\rangle_1\,]\,. \tag{48}$$

From Eq. (1.5) of the text, we get

$$\tfrac{1}{2}\hbar\psi\gamma_3\widetilde{\psi} = \rho s\,, \tag{49}$$

from which we get

$$i\rho s = \tfrac{1}{2}\hbar\psi i\gamma_3\widetilde{\psi}\,. \tag{50}$$

With this result, (48) becomes (by use of the product rule, which works so simply because ∂_μ is a scalar operator):

$$\partial_\mu(i\rho s) = \frac{\hbar}{2}[\,\partial_\mu\psi i\gamma_3\widetilde{\psi} + \psi i\gamma_3\partial_\mu\widetilde{\psi}\,]\,, \tag{51}$$

which is Eq. (2.13). It's important to note that to reduce the number of parentheses in a given expression, Hestenes has apparently followed the rule that the action of a derivative on an expression extends only to the variable closest to it on its right. Therefore, we interpret

$$\partial_\mu \psi i\gamma_3 \widetilde{\psi} \quad \text{as} \quad (\partial_\mu \psi) i\gamma_3 \widetilde{\psi}\,. \tag{52}$$

I sometimes make this explicit, which is why my use of parentheses is a bit different than that of Hestenes.

On multiplying (51) through by γ^μ, we get

$$\begin{aligned}\square(i\rho s) &= \frac{\hbar}{2}[\,(\square\psi)i\gamma_3\widetilde{\psi} + \gamma^\mu\psi i\gamma_3\partial_\mu\widetilde{\psi}\,] \\ &= \frac{\hbar}{2}[\,(\square\psi)i\gamma_3\widetilde{\psi} + \gamma^\mu\langle\,\psi i\gamma_3\partial_\mu\widetilde{\psi}\,\rangle_1\,] \\ &= \frac{\hbar}{2}[\,(\square\psi)i\gamma_3\widetilde{\psi} + \gamma^\mu\langle\,\psi i\gamma_3\partial_\mu\widetilde{\psi}\,\rangle_1^{\sim}\,] \\ &= \frac{\hbar}{2}[\,(\square\psi)i\gamma_3\widetilde{\psi} - \gamma^\mu\langle\,\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1\,]\,,\end{aligned} \tag{53}$$

which is off by a factor of 2. Obviously, I need more work on this to get the text's result:

$$\hbar(\square\psi)i\gamma_3\widetilde{\psi} = \hbar\gamma^\mu\langle\,\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 + \square(i\rho s)\,, \tag{54}$$

which is Eq. (2.14).

As presented in Ref. [5], the Dirac equation is

$$\hbar\square\psi i\gamma_3\gamma_0 = m\psi\gamma_0 + eA\psi\,, \tag{55}$$

which is (2.15). Next, we multiply on the right by $\gamma_0\widetilde{\psi}$, remembering that $\psi\widetilde{\psi} = \rho e^{i\beta}$, to get Eq. (2.16):

$$\hbar\square\psi i\gamma_3\widetilde{\psi} = m\rho e^{i\beta} + eA\rho v\,, \tag{56}$$

where I am interpreting the LHS as meaning

$$\hbar\square\psi i\gamma_3\widetilde{\psi} = \hbar(\square\psi)i\gamma_3\widetilde{\psi}\,. \tag{57}$$

To modify this last result to get the next one, we need the following facts:

$$\rho p = v^\mu T_\mu\,, \tag{58a}$$

$$\bar{T}_\mu = \hbar\langle\,\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 - e\rho v A_\mu\,, \tag{58b}$$

$$\hbar(\square\psi)i\gamma_3\widetilde{\psi} = \hbar\gamma^\mu\langle\,\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 + \square(i\rho s)\,. \tag{58c}$$

We begin by multiplying (58b) on the left by γ^μ:

$$\gamma^\mu\bar{T}_\mu = \hbar\gamma^\mu\langle\,\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 - eA\rho v\,. \tag{59}$$

Next, we use (58c) to get

$$\gamma^\mu\bar{T}_\mu = \hbar(\square\psi)i\gamma_3\widetilde{\psi} - \square(i\rho s) - eA\rho v\,. \tag{60}$$

Then we use (56) to get Eq. (2.17):

$$\gamma^\mu \bar{T}_\mu = m\rho e^{i\beta} - \Box(i\rho s) = m\rho e^{i\beta} + i\Box(\rho s)\,, \tag{61}$$

which is true because i anticommutes with all vectors, and $\Box$ act to i as just a vector.

To find the pseudoscalar part of (61), we first expand it:

$$\gamma^\mu \bar{T}_\mu = m\rho(\cos\beta + i\sin\beta) + i\Box\cdot(\rho s) + i\Box\wedge(\rho s)\,. \tag{62}$$

Thus, the pseudoscalar part is

$$0 = m\rho\sin\beta + \Box\cdot(\rho s)\,, \tag{63}$$

which gives us Eq. (2.18)

$$\Box\cdot(\rho s) = -m\rho\sin\beta\,. \tag{64}$$

The trace of the Tetrode tensor T^μ_μ comes from the scalar part of (62):

$$\gamma^\mu\cdot T_\mu = T_\mu\cdot\gamma^\mu = m\rho\cos\beta\,, \tag{65}$$

which is (2.19). The bivector part of (61) yields (2.20)

$$\gamma^\mu\wedge\bar{T}_\mu = T_\mu\wedge\gamma^\mu = i(\Box\wedge\rho s) = -\Box\cdot(i\rho s)\,. \tag{66}$$

Let me demonstrate this reasoning. Let a be a vector, then

$$\begin{aligned} i(\Box\wedge a) &= \langle\, i\Box\wedge a\,\rangle_2 \\ &= \langle\, i\Box a\,\rangle_2 \\ &= \langle\, -\Box i a\,\rangle_2 \\ &= -\Box\cdot(ia)\,. \end{aligned} \tag{67}$$

Changing the dummy indices of summation in (62), we can write

$$T_\beta\wedge\gamma^\beta = (\gamma^\alpha\wedge\gamma^\beta)T_{\beta\alpha}\,. \tag{68}$$

The easiest way to prove this is to start on the RHS and derive the LHS using (38).

$$(\gamma^\alpha\wedge\gamma^\beta)T_{\beta\alpha} = (T_{\beta\alpha}\gamma^\alpha\wedge\gamma^\beta) = T_\beta\wedge\gamma^\beta\,. \tag{69}$$

On multiplying by an antisymmetric operator:

$$(\gamma_\mu\wedge\gamma_\nu)\cdot T_\beta\wedge\gamma^\beta = (\gamma_\mu\wedge\gamma_\nu)\cdot(\gamma^\alpha\wedge\gamma^\beta)T_{\beta\alpha} = T_{\mu\nu} - T_{\nu\mu}\,, \tag{70}$$

where we used the identity

$$(\gamma_\mu\wedge\gamma_\nu)\cdot(\gamma^\alpha\wedge\gamma^\beta) = \delta^\alpha_\nu\delta^\beta_\mu - \delta^\alpha_\mu\delta^\beta_\nu\,. \tag{71}$$

Hence, the first part of (2.21):

$$(\gamma_\mu \wedge \gamma_\nu) \cdot (\gamma^\alpha \wedge \gamma^\beta) T_{\beta\alpha} = (\delta^\alpha_\nu \delta^\beta_\mu - \delta^\alpha_\mu \delta^\beta_\nu) T_{\beta\alpha} = T_{\mu\nu} - T_{\nu\mu}\,. \tag{72}$$

Now we refer back to (70) and get the second part of (2.21):

$$(\gamma_\mu \wedge \gamma_\nu) \cdot [\, i(\Box \wedge \rho s)\,] = i\gamma_\mu \wedge \gamma_\nu \wedge \Box \wedge (\rho s) = -\epsilon_{\mu\nu\alpha\beta}\partial^\alpha(\rho s^\beta)\,. \tag{73}$$

Going back to (58b) and taking the partial by μ, we get Eq. (2.23):

$$\begin{aligned} \partial_\mu \bar{T}^\mu &= \hbar \partial_\mu \langle\, (\partial^\mu \psi) i\gamma_3 \widetilde{\psi}\,\rangle_1 - e\partial_\mu(\rho v A^\mu) \\ &= \hbar \langle\, (\Box^2 \psi) i\gamma_3 \widetilde{\psi}\,\rangle_1 - e\partial_\mu(\rho v A^\mu)\,, \end{aligned} \tag{74}$$

where, of course, $\Box^2 = \partial_\mu \partial^\mu$. Okay, so now we differentiate T^μ, which we get from (40) in the raised form as

$$T^\mu = \rho v^\mu p + N^\mu\,. \tag{75}$$

Then,

$$\partial_\mu T^\mu = \partial_\mu(\rho v^\mu p) + \partial_\mu N^\mu\,. \tag{76}$$

Equation (2.22) claims that

$$\partial_\mu T^\mu = \partial_\mu \bar{T}^\mu\,, \tag{77}$$

but I don't at this time have a proof for this.

Now, we wish to express the first term on the RHS of (74) in terms of observables. First, we take the gradient of (55):

$$\hbar \Box^2 \psi i\gamma_3\gamma_0 = m\Box\psi\gamma_0 + e\Box A\psi\,. \tag{78}$$

Next, we multiply on the right by $\gamma_0\widetilde{\psi}$:

$$\hbar(\Box^2\psi) i\gamma_3 \widetilde{\psi} = m(\Box\psi)\widetilde{\psi} + e(\Box A\psi)\gamma_0\widetilde{\psi}\,. \tag{79}$$

Now, we try to get Eq. (2.24), which is

$$\hbar\Box^2\psi i\gamma_3\widetilde{\psi} = \hbar^{-1}(e^2A^2 - m^2)\, i\rho s + e(\Box A)\rho v + 2e(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,. \tag{80}$$

To arrive at this last equation from the equation before it, we need to do two things. The first is to expand the expession $\Box A\psi$. The second thing is to 'remove' all gradients of ψ; that is, expressions of the form $\Box\psi$. We can look to the Dirac equation for the suitable expression to use to replace it with.

We begin by expanding $\Box A\psi$

$$\begin{aligned} \Box A\psi &= \dot{\Box}\dot{A}\dot{\psi} \\ &= (\Box A)\psi + (\dot{\Box}A\dot{\psi}) \\ &= (\Box A)\psi + (2A\cdot\Box\psi - A\Box\psi) \\ &= (\Box A)\psi + 2(A\cdot\Box\psi) - (A\Box\psi)\,. \end{aligned} \tag{81}$$

So, the second term on the RHS of (79) becomes

$$
\begin{aligned}
e(\Box A\psi)\gamma_0\widetilde{\psi} &= e[\,(\Box A)\psi + 2(A\cdot\Box\psi) - (A\Box\psi)\,]\gamma_0\widetilde{\psi} \\
&= e[\,(\Box A)\psi\gamma_0\widetilde{\psi} + 2(A\cdot\Box\psi)\gamma_0\widetilde{\psi} - (A\Box\psi)\gamma_0\widetilde{\psi}\,] \\
&= e[\,(\Box A)\rho v + 2(A\cdot\Box\psi)\gamma_0\widetilde{\psi} - (A\Box\psi)\gamma_0\widetilde{\psi}\,]\,. \qquad (82)
\end{aligned}
$$

Substituting this last result into (79), we have that

$$
\begin{aligned}
\hbar(\Box^2\psi)i\gamma_3\widetilde{\psi} &= m(\Box\psi)\widetilde{\psi} + e[\,(\Box A)\rho v + 2(A\cdot\Box\psi)\gamma_0\widetilde{\psi} - (A\Box\psi)\gamma_0\widetilde{\psi}\,] \\
&= m(\Box\psi)\widetilde{\psi} - e(A\Box\psi)\gamma_0\widetilde{\psi} + e(\Box A)\rho v + 2e(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,. \qquad (83)
\end{aligned}
$$

Let's now restate the Dirac equation and then solve it for $\Box\psi$.

$$
\hbar\Box\psi i\gamma_3\gamma_0 = m\psi\gamma_0 + eA\psi\,. \qquad (84)
$$

First, we multiply through by $\hbar^{-1}$:

$$
\Box\psi i\gamma_3\gamma_0 = \hbar^{-1}[\,m\psi\gamma_0 + eA\psi\,]\,. \qquad (85)
$$

Next, we multiply through on the right by $\gamma_0\gamma_3 i$, to get

$$
\begin{aligned}
\Box\psi &= \hbar^{-1}[\,m\psi\gamma_0 + eA\psi\,]\gamma_0\gamma_3 i \\
&= \hbar^{-1}[\,m\psi\gamma_3 i + eA(\psi\gamma_0\gamma_3 i)\,]\,. \qquad (86)
\end{aligned}
$$

Thus, $m(\Box\psi)\widetilde{\psi}$ becomes

$$
\begin{aligned}
m(\Box\psi)\widetilde{\psi} &= m\hbar^{-1}[\,m\psi\gamma_3 i\widetilde{\psi} + eA\psi\gamma_0\gamma_3 i\widetilde{\psi}\,] \\
&= m\hbar^{-1}[\,-mi\psi\gamma_3\widetilde{\psi} + eAi\psi\gamma_0\gamma_3\widetilde{\psi}\,] \\
&= m\hbar^{-1}[\,-mi\rho s + eAi\psi\gamma_0\gamma_3\widetilde{\psi}\,]\,. \qquad (87)
\end{aligned}
$$

Now, we do similarly to $e(A\Box\psi)\gamma_0\widetilde{\psi}$:

$$
\begin{aligned}
e(A\Box\psi)\gamma_0\widetilde{\psi} &= e\hbar^{-1}A[\,m\psi\gamma_3 i + e(\psi\gamma_0\gamma_3 i)\,]\gamma_0\widetilde{\psi} \\
&= e\hbar^{-1}A[\,m\psi\gamma_3 i + e(\psi\gamma_0\gamma_3 i)\,]\gamma_0\widetilde{\psi} \\
&= e\hbar^{-1}A[\,-mi\psi\gamma_3\gamma_0\widetilde{\psi} - eAi\psi\gamma_3\widetilde{\psi}\,] \\
&= e\hbar^{-1}A[\,-mi\psi\gamma_3\gamma_0\widetilde{\psi} - eAi\rho s\,]\,. \qquad (88)
\end{aligned}
$$

Substituting these into (83), we get

$$
\begin{aligned}
\hbar(\Box^2\psi)i\gamma_3\widetilde{\psi} &= m\hbar^{-1}[\,-mi\rho s + eAi\psi\gamma_0\gamma_3\widetilde{\psi}\,] - e\hbar^{-1}A[\,-mi\psi\gamma_3\gamma_0\widetilde{\psi} - eAi\rho s\,] \\
&\quad + e[\,(\Box A)\rho v + 2(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,] \\
&= \hbar^{-1}[\,-m^2 i\rho s + emAi\psi\gamma_0\gamma_3\widetilde{\psi}\,] - \hbar^{-1}[\,meAi\psi\gamma_0\gamma_3\widetilde{\psi} - e^2A^2 i\rho s\,] \\
&\quad + e[\,(\Box A)\rho v + 2(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,] \\
&= \hbar^{-1}[\,-m^2 i\rho s + e^2A^2 i\rho s\,] + e(\Box A)\rho v + 2e(A\cdot\Box\psi)\gamma_0\widetilde{\psi} \\
&= \hbar^{-1}(e^2A^2 - m^2)2i\rho s + e(\Box A)\rho v + 2e(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,, \qquad (89)
\end{aligned}
$$

where the two terms involving $i\psi\gamma_0\gamma_3\widetilde{\psi}$ cancelled each other. And this is Eq. (2.24).

Going on. We want the vector part of (89). We begin with the identity

$$\langle\, e(\Box A)\rho v\,\rangle_1 = e\rho(\Box\wedge A)\cdot v + e(\Box\cdot A)\rho v\,. \tag{90}$$

So, the vector part of (89) is

$$\hbar\langle\,\Box^2\psi i\gamma_3\widetilde{\psi}\,\rangle_1 = e\rho(\Box\wedge A)\cdot v + e(\Box\cdot A)\rho v + 2e\langle\,(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,\rangle_1\,. \tag{91}$$

and we need a more elegant expression for the last term. We start with the familiar equation

$$\psi\gamma_0\widetilde{\psi} = \rho v\,. \tag{92}$$

On differentiating this by $A\cdot\Box$, we get

$$(A\cdot\Box\psi)\gamma_0\widetilde{\psi} + \psi\gamma_0(A\cdot\Box)\widetilde{\psi} = A\cdot\Box(\rho v)\,. \tag{93}$$

But $(A\cdot\Box\psi)\gamma_0\widetilde{\psi} = \psi\gamma_0(A\cdot\Box)\widetilde{\psi}$, so

$$2\langle\,(A\cdot\Box\psi)\gamma_0\widetilde{\psi}\,\rangle_1 = A\cdot\Box(\rho v)\,. \tag{94}$$

Therefore, (91) becomes

$$\hbar\langle\,\Box^2\psi i\gamma_3\widetilde{\psi}\,\rangle_1 = e\rho(\Box\wedge A)\cdot v + e(\Box\cdot A)\rho v + eA\cdot\Box(\rho v)\,, \tag{95}$$

which is the second result of (2.25). But this equation can be put into a simpler form, beginning with the fact that $\Box\wedge A = F$. We can also perform some 'tensor' operations:

$$\begin{aligned} e\partial_\mu(\rho v A^\mu) &= e\partial_\mu(\rho v)A^\mu + e\rho v\partial_\mu A^\mu \\ &= eA\cdot\Box(\rho v) + e(\Box\cdot A)\rho v\,. \end{aligned} \tag{96}$$

Hence, (95) becomes

$$\hbar\langle\,\Box^2\psi i\gamma_3\widetilde{\psi}\,\rangle_1 = e\rho F\cdot v + e\partial_\mu(\rho v A^\mu)\,, \tag{97}$$

which is (2.25).

Now, defining $f = F\cdot v$ and using (74) and (77), we get

$$\partial_\mu T^\mu = \rho e F\cdot v = \rho f\,. \tag{98}$$

Then, coupling this with (40), we get part of Eq. (2.26):

$$\begin{aligned} \partial_\mu T^\mu &= \partial_\mu(\rho v^\mu p) + \partial_\mu N^\mu \\ &= \partial_\mu(\rho v^\mu)p + \rho v^\mu\partial_\mu p + \partial_\mu N^\mu \\ &= \rho v\cdot\Box p + \partial_\mu N^\mu \\ &= \rho\dot{p} + \partial_\mu N^\mu = \rho e F\cdot v = \rho f\,, \end{aligned} \tag{99}$$

where we have assumed that $\partial_\mu(\rho v^\mu) = 0$. But this is easy to prove using tensors. First, recall that

$$\Box \cdot (\rho v) = 0\,. \tag{100}$$

So that

$$\begin{aligned}\Box \cdot (\rho v) &= \gamma^\nu \partial_\nu \cdot (\rho v^\mu \gamma_\mu) \\ &= \gamma^\nu \cdot \gamma_\mu \partial_\nu (\rho v^\mu) \\ &= \delta^\nu_\mu \partial_\nu \,(\rho v^\mu) \\ &= \partial_\mu \,(\rho v^\mu) = 0\,. \end{aligned} \tag{101}$$

Equation (2.27) is

$$\begin{aligned}\partial_\mu (T^\mu \wedge x) &= \partial_\mu (T^\mu \wedge x) + \partial_\mu (T^\mu \wedge x) \\ &= \rho f \wedge x + T^\mu \wedge \gamma_\mu\,. \end{aligned} \tag{102}$$

With help from (66) we have Eq. (2.28):

$$T_\mu \wedge \gamma^\mu = -\partial_\mu S^\mu\,. \tag{103}$$

For the next equation, we remember that $S = isv$.

$$S^\mu = \rho is \wedge \gamma^\mu = \rho(is) \cdot \gamma^\mu = \rho(S \wedge v) \cdot \gamma^\mu\,. \tag{104}$$

Now,

$$\begin{aligned}(S \wedge v) \cdot \gamma^\mu &= \langle\, Sv\gamma^\mu \,\rangle_2 = -\langle\, Sv\gamma^\mu \,\rangle_2^{\sim} \\ &= -\langle\, \gamma^\mu v \widetilde{S} \,\rangle_2 = \langle\, \gamma^\mu v S \,\rangle_2 \\ &= \gamma^\mu \cdot (v \wedge S) = \gamma^\mu \cdot v S + v \gamma^\mu \cdot S \\ &= v^\mu S + S \cdot \gamma^\mu v\,. \end{aligned} \tag{105}$$

Thus, Eq. (2.29) is

$$S^\mu = \rho v^\mu S + \rho S \cdot \gamma^\mu v\,. \tag{106}$$

If we define the vector J^μ as in Eq. (2.30):

$$J^\mu \equiv T^\mu \wedge x + S^\mu\,, \tag{107}$$

then, using (103), we can rewrite (102) as

$$\partial_\mu J^\mu = \rho f \wedge x\,, \tag{108}$$

which is (2.31). The proper angular momentum density is given as

$$J(v) = v_\mu J^\mu = v_\mu T^\mu \wedge x + v_\mu S^\mu\,, \tag{109}$$

which is almost (2.32). For the first term on the RHS, we have that

$$v_\mu T^\mu \wedge x = \rho p\,, \tag{110}$$

which used (39). Now, using that

$$S = \rho^{-1} v_\mu S^\mu = i s \wedge v\,, \tag{111}$$

then, finally, (2.32):

$$J(v) = \rho(p \wedge x + S)\,. \tag{112}$$

Remembering that $v_\mu v^\mu = 1$, for the second term we have

$$\begin{aligned} v_\mu S^\mu &= \rho v_\mu v^\mu S + \rho S \cdot (v_\mu \gamma^\mu) v \\ &= \rho S + \rho S \cdot v v \\ &= \rho S\,, \end{aligned} \tag{113}$$

since $S \cdot v = 0$.

Using (103) and (107) we get

$$\begin{aligned} \partial_\mu J^\mu &= \partial_\mu (T^\mu \wedge x) + \partial_\mu S^\mu \\ &= (\partial_\mu T^\mu) \wedge x + T^\mu \wedge \gamma_\mu + \partial_\mu S^\mu \\ &= (\partial_\mu T^\mu) \wedge x\,. \end{aligned} \tag{114}$$

Defining M^μ as

$$M^\mu \equiv \rho S \cdot \gamma^\mu v\,, \tag{115}$$

which is Eq. (2.33). Then

$$\partial_\mu M^\mu = \partial_\mu (\rho S \cdot \gamma^\mu v)\,. \tag{116}$$

All this can be combined to yield Eq. (2.34):

$$\rho \dot{S} + \rho p \wedge v = \gamma_\mu \wedge N^\mu - \partial_\mu M^\mu\,, \tag{117}$$

and $\dot{S} = v \cdot \Box S$. (By the way, we have corrected here an error in the preprint.)

3 Local Momentum and angular velocity

Starting with

$$e_\alpha = R \gamma_\alpha \widetilde{R}\,, \tag{118}$$

we get Eq. (3.1):

$$\gamma_\mu \cdot \Box e_\alpha = \partial_\mu e_\alpha = \tfrac{1}{2} [\, \Omega_\mu, e_\alpha \,] = \Omega_\mu \cdot e_\alpha\,, \tag{119}$$

where

$$\Omega_\mu \equiv 2 (\partial_\mu R) \widetilde{R}\,. \tag{120}$$

Proof:

Let's begin with $R\widetilde{R} = 1$ and differentiate it by $\partial_\mu = \gamma_\mu \cdot \Box$, to get

$$(\partial_\mu R)\widetilde{R} = -R\partial_\mu \widetilde{R}\,. \tag{121}$$

By first taking the reverse on both sides of this, we can then solve it for $\partial_\mu \widetilde{R}$, to get

$$\partial_\mu \widetilde{R} = -\widetilde{R}(\partial_\mu R)\widetilde{R}\,. \tag{122}$$

Therefore

$$\begin{aligned}\gamma_\mu \cdot \Box\, e_\alpha &= \partial_\mu e_\alpha = \partial_\mu R \gamma_\alpha \widetilde{R} \\ &= (\partial_\mu R)\gamma_\alpha \widetilde{R} + R\gamma_\alpha \partial_\mu \widetilde{R} \\ &= (\partial_\mu R)\widetilde{R} R \gamma_\alpha \widetilde{R} - R\gamma_\alpha \widetilde{R}(\partial_\mu R)\widetilde{R} \\ &= (\partial_\mu R)\widetilde{R} e_\alpha - e_\alpha (\partial_\mu R)\widetilde{R} \\ &= \tfrac{1}{2}\Omega_\mu e_\alpha - \tfrac{1}{2} e_\alpha \Omega_\mu \\ &= \tfrac{1}{2}[\,\Omega_\mu, e_\alpha\,] = \Omega_\mu \cdot e_\alpha\,. \end{aligned} \tag{123}$$

The frame of e_μ rotates with angular velocity Ω_μ. To make this meaningful, we must express this rotation in terms of local variables. To that end, define P_μ and q_μ according as

$$P_\mu + i q_\mu = \frac{\hbar}{2}(\partial_\mu R \gamma_2 \gamma_1 \widetilde{R} - R\gamma_2\gamma_1 \partial_\mu \widetilde{R})\,, \tag{124}$$

which is Eq. (3.3). Remembering that

$$S \equiv i s v = \frac{\hbar}{2} e_2 e_1 = \frac{\hbar}{2} R\gamma_2\gamma_1 \widetilde{R}\,, \tag{125}$$

then

$$\partial_\mu S = \frac{\hbar}{2}(\partial_\mu R \gamma_2 \gamma_1 \widetilde{R} + R\gamma_2\gamma_1 \partial_\mu \widetilde{R})\,. \tag{126}$$

We also get (first by virtual emplacement of $1 = \widetilde{R}R$ and then by using the recent definitions) that

$$\hbar(\partial_\mu R)\gamma_2\gamma_1 \widetilde{R} = [\,(2\partial_\mu R)\widetilde{R}\,](\frac{\hbar}{2} R\gamma_2\gamma_1\widetilde{R}) = \Omega_\mu S\,, \tag{127}$$

which is Eq. (3.5). Adding (124) and (126), we get (3.6):

$$P_\mu + i q_\mu + \partial_\mu S = \Omega_\mu S = \hbar \partial_\mu R \gamma_2\gamma_1 \widetilde{R}\,. \tag{128}$$

Taking the scalar part of this, we get Eq. (3.7):

$$P_\mu = \Omega_\mu \cdot S = \hbar \langle\, \partial_\mu R \gamma_2 \gamma_1 \widetilde{R}\,\rangle\,. \tag{129}$$

To get at the pseudoscalar part of (128), we may multiply through by $-i$ and then take the scalar part of the result [Eq. (3.8)]:

$$q_\mu = -i\Omega_\mu \wedge S = \hbar \langle\, (-i)\partial_\mu R \gamma_2\gamma_1 \widetilde{R}\,\rangle\,, \tag{130}$$

which corrects a misprint in the preprint and a couple in the published article. Last, but not least, is the bivector part of (128):

$$\partial_\mu S = \Omega_\mu \times S\,. \tag{131}$$

Adding in some identitites, we get

$$\partial_\mu S = \Omega_\mu \times S = \tfrac{1}{2}(\Omega_\mu S - S\Omega_\mu) = \tfrac{1}{2}[\,\Omega_\mu, S\,]\,, \tag{132}$$

which is Eq. (3.9)

Now, we solve for Ω_μ, using (128), we get (3.10):

$$\begin{aligned}\Omega_\mu &= (\partial_\mu S + P_\mu + iq_\mu)S^{-1} \\ &= s(\partial_\mu v)vs^{-1} + (\partial_\mu s)s^{-1} + q_\mu vs^{-1} + P_\mu S^{-1}\,,\end{aligned} \tag{133}$$

where $S^{-1} = |\,S\,|^{-2}\widetilde{S} = is^{-1}v$, $s^{-1} = -|\,s\,|^{-2}s$.

Proof: For my proof, I use that $S = ivs$, and $S^{-1} = -vs^{-1}i$. Then I just take the partial derivative:

$$\partial_\mu S = \partial_\mu(isv) = i[\,(\partial_\mu s)v + s(\partial_\mu v)\,]\,. \tag{134}$$

Therefore

$$(\partial_\mu S)S^{-1} = (\partial_\mu s)s^{-1} + s(\partial_\mu v)vs^{-1}\,. \tag{135}$$

So, (136) gives us

$$\begin{aligned}\Omega_\mu &= (\partial_\mu S)S^{-1} + P_\mu S^{-1} + iq_\mu S^{-1} \\ &= s(\partial_\mu v)vs^{-1} + (\partial_\mu s)s^{-1} + q_\mu vs^{-1} + P_\mu S^{-1}\,,\end{aligned} \tag{136}$$

which is Eq. (3.10). Next, we need to re-express q_μ. We begin with

$$\partial_\mu v = \Omega_\mu \cdot v\,. \tag{137}$$

On dotting (136) by v, we get (where $v \cdot s = 0$ implies that $v \cdot s^{-1} = 0$)

$$\begin{aligned}\Omega_\mu \cdot v &= [\,s(\partial_\mu v)vs^{-1} + (\partial_\mu s)s^{-1} + q_\mu vs^{-1} + P_\mu S^{-1}\,]\cdot v \\ &= [\,s(\partial_\mu v)vs^{-1} + (\partial_\mu s)s^{-1} + q_\mu vs^{-1}\,]\cdot v \\ &= [\,s(\partial_\mu v)vs^{-1}\,]\cdot v + [\,(\partial_\mu s)s^{-1}\,]\cdot v + q_\mu[\,v \wedge s^{-1}\,]\cdot v \\ &= [\,s(\partial_\mu v)vs^{-1}\,]\cdot v + [\,(\partial_\mu s) \wedge s^{-1}\,]\cdot v - q_\mu s^{-1} \\ &= [\,s(\partial_\mu v)vs^{-1}\,]\cdot v - s^{-1}v\cdot(\partial_\mu s) - q_\mu s^{-1}\,.\end{aligned} \tag{138}$$

Finally, to the first term. (Note that for vectors a, b, $ab = 2a\cdot b - ba$.)

$$\begin{aligned}[\,s(\partial_\mu v)vs^{-1}\,]\cdot v &= [\,s(\partial_\mu v)vs^{-1}\,]\cdot v \\ &= [\,\{2s\cdot(\partial_\mu v) - (\partial_\mu v)s\}vs^{-1}\,]\cdot v \\ &= [\,2s\cdot(\partial_\mu v)vs^{-1}\,]\cdot v - [\,(\partial_\mu v)svs^{-1}\,]\cdot v \\ &= 2s\cdot(\partial_\mu v)\,v\wedge s^{-1}\cdot v + [\,(\partial_\mu v)ss^{-1}v\,]\cdot v \\ &= 2s\cdot(\partial_\mu v)(-s^{-1}) + [\,(\partial_\mu v)\wedge v\,]\cdot v \\ &= vs\cdot(\partial_\mu s)s^{-1} + \partial_\mu v - q_\mu s^{-1}\,.\end{aligned} \tag{139}$$

where, owing to the fact that $v^2 = v \cdot v = 1, v \cdot (\partial_\mu v) = 0$. So, plugging this last result into (138), we get

$$\partial_\mu v = \Omega_\mu \cdot v = s^{-1}(v \cdot \partial_\mu s) + \partial_\mu v - q_\mu s^{-1} . \tag{140}$$

On dropping the $\partial_\mu v$ term from both sides, we can solve for q_μ, to get

$$q_\mu = v \cdot \partial_\mu s , \tag{141}$$

which is Eq. (3.11). With $q = \gamma^\mu q_\mu$, we get (3.12):

$$q = \gamma^\mu v \cdot \partial_\mu s . \tag{142}$$

But because $v \cdot s = 0$, then

$$\partial_\mu (v \cdot s) = 0 , \tag{143}$$

from which we get that

$$v \cdot \partial_\mu s = -s \cdot \partial_\mu v . \tag{144}$$

Hence,

$$q = \gamma^\mu v \cdot \partial_\mu s = \gamma^\mu \dot{\partial}_\mu v \cdot \dot{s} = \dot{\Box} v \cdot \dot{s} . \tag{145}$$

Now, for another identity:

$$v \cdot (\Box \wedge s) = v \cdot \Box s - \dot{\Box} v \cdot \dot{s} , \tag{146}$$

From this we get that

$$\dot{\Box} v \cdot \dot{s} = -v \cdot (\Box \wedge s) + v \cdot \Box s . \tag{147}$$

Therefore, (145) becomes

$$q = \gamma^\mu v \cdot \partial_\mu s = -v \cdot (\Box \wedge s) + v \cdot \Box s , \tag{148}$$

which is Eq. (3.12).

Next is Eq. (3.13):

$$P_\mu = -\frac{\hbar}{2} e_2 \cdot \partial_\mu e_1 = \frac{\hbar}{2} e_1 \cdot \partial_\mu e_2 . \tag{149}$$

Let's begin with the easy part. Since $e_2 \cdot e_1 = 0$ then $-e_2 \cdot \partial_\mu e_1 = e_1 \cdot \partial_\mu e_2$. Now, since $e_2 = \langle R \gamma_2 \widetilde{R} \rangle_1$, then

$$\partial_\mu e_2 = \langle (\partial_\mu R) \gamma_2 \widetilde{R} \rangle_1 + \langle R \gamma_2 \partial_\mu \widetilde{R} \rangle_1 . \tag{150}$$

Then,

$$
\begin{aligned}
e_1 \cdot \partial_\mu e_2 &= \langle\, e_1(\partial_\mu R)\gamma_2 \widetilde{R}\,\rangle + \langle\, e_1 R\gamma_2 \partial_\mu \widetilde{R}\,\rangle \\
&= \langle\, e_1(\partial_\mu R)\gamma_2 \widetilde{R}\,\rangle + \langle\, e_1 R\gamma_2 \partial_\mu \widetilde{R}\,\rangle^{\sim} \\
&= \langle\, e_1(\partial_\mu R)\gamma_2 \widetilde{R}\,\rangle + \langle\, (\partial_\mu R)\gamma_2 \widetilde{R} e_1\,\rangle \\
&= \langle\, e_1(\partial_\mu R)\gamma_2 \widetilde{R}\,\rangle + \langle\, e_1(\partial_\mu R)\gamma_2 \widetilde{R}\,\rangle \\
&= 2\langle\, e_1(\partial_\mu R)\gamma_2 \widetilde{R}\,\rangle \\
&= 2\langle\, (\partial_\mu R)\gamma_2 \widetilde{R} e_1\,\rangle \\
&= 2\langle\, (\partial_\mu R)\gamma_2 \widetilde{R}(R\gamma_1\widetilde{R})\,\rangle \\
&= 2\langle\, (\partial_\mu R)\gamma_2\gamma_1 \widetilde{R}\,\rangle \\
&= 2\hbar^{-1} P_\mu\,.
\end{aligned} \tag{151}
$$

On solving this for P_μ, we get (149).

What happens to Ω_μ if we make a phase change on R according as

$$
R \longrightarrow R e^{-\gamma_2\gamma_1\Lambda/\hbar}\,. \tag{152}
$$

$$
\Omega_\mu \longrightarrow \Omega_\mu + \partial_\mu \Lambda S^{-1}\,. \tag{153}
$$

And from (120) we have Eq. (3.14):

$$
P_\mu \longrightarrow P_\mu + \partial_\mu \Lambda\,. \tag{154}
$$

The calculation is straightforward this time. [Note: $\Omega_\mu = 2(\partial_\mu R)\widetilde{R}$]

$$
\begin{aligned}
\Omega'_\mu &= 2(\partial_\mu R e^{-\gamma_2\gamma_1\Lambda/\hbar}) e^{\gamma_2\gamma_1\Lambda/\hbar}\widetilde{R} \\
&= 2[\,(\partial_\mu R)e^{-\gamma_2\gamma_1\Lambda/\hbar} + R(-\frac{\gamma_2\gamma_1}{\hbar}(\partial_\mu\Lambda))\,]e^{\gamma_2\gamma_1\Lambda/\hbar}\widetilde{R} \\
&= 2(\partial_\mu R)\widetilde{R} - 2R[\,\frac{\gamma_2\gamma_1}{\hbar}(\partial_\mu\Lambda)\,]\widetilde{R} \\
&= 2(\partial_\mu R)\widetilde{R} - 2(\partial_\mu\Lambda)R\frac{\gamma_2\gamma_1}{\hbar}\widetilde{R} \\
&= \Omega_\mu + (\partial_\mu\Lambda)S^{-1}\,.
\end{aligned} \tag{155}
$$

Now, we use (1) and (120) to get Eq. (3.15):

$$
\begin{aligned}
\partial_\mu\psi &= \partial_\mu[\,\rho^{1/2}e^{i\beta}R\,] \\
&= \tfrac{1}{2}\rho^{-1/2}(\partial_\mu\rho)e^{i\beta}R + \rho^{1/2}i(\partial_\mu\beta)e^{i\beta}R + \rho^{1/2}e^{i\beta}\partial_\mu R \\
&= \tfrac{1}{2}(\partial_\mu \ln\rho)\rho^{1/2}e^{i\beta}R + \rho^{1/2}i(\partial_\mu\beta)e^{i\beta}R + \rho^{1/2}e^{i\beta}R(\widetilde{R}\partial_\mu R) \\
&= \tfrac{1}{2}(\partial_\mu \ln\rho)\psi + i(\partial_\mu\beta)\psi + \psi(\partial_\mu R)\widetilde{R} \\
&= \tfrac{1}{2}(\partial_\mu \ln\rho)\psi + i(\partial_\mu\beta)\psi + \tfrac{1}{2}\psi 2(\partial_\mu R)\widetilde{R} \\
&= \tfrac{1}{2}(\partial_\mu \ln\rho)\psi + i(\partial_\mu\beta)\psi + \tfrac{1}{2}\psi\Omega_\mu \\
&= \tfrac{1}{2}[\,\partial_\mu(\ln\rho e^{i\beta}) + \Omega_\mu\,]\psi\,.
\end{aligned} \tag{156}
$$

Now we multiply through on the right by $\hbar i\gamma_3\widetilde{\psi}$ to get Eq. (3.16):

$$\begin{aligned}\hbar(\partial_\mu\psi)i\gamma_3\widetilde{\psi} &= \tfrac{1}{2}[\,\partial_\mu(\ln\rho e^{i\beta}) + \Omega_\mu\,]\psi\hbar i\gamma_3\widetilde{\psi} \\ &= \tfrac{1}{2}[\,\partial_\mu(\ln\rho e^{i\beta}) + \Omega_\mu\,]\rho S v \\ &= [\,P_\mu + iq_\mu + W_\mu\,]\rho v\,, \end{aligned} \tag{157}$$

where we used that $\frac{1}{2}\hbar\psi i\gamma_3\widetilde{\psi} = \rho S v$ and

$$W_\mu = (\rho e^{i\beta})^{-1}\partial_\mu(\rho e^{i\beta}S) = \partial_\mu S + S(\partial_\mu \ln\rho + i\partial_\mu\beta)\,. \tag{158}$$

On extracting the vector part of (157), we get (3.18):

$$\langle\,\hbar\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 = \rho(vP_\mu - v\cdot W_\mu)\,, \tag{159}$$

from which we get that

$$\begin{aligned}\hbar\gamma_\nu\cdot\langle\,\partial_\mu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 &= \rho\big(\gamma_\nu\cdot vP_\mu - \gamma_\nu\cdot(v\cdot W_\mu)\big) \\ &= \rho\big(v_\nu P_\mu + (v\wedge\gamma_\nu)\cdot W_\mu\big)\,, \end{aligned} \tag{160}$$

which is (3.19), and this corrects a mistake that's in both the preprint and the published article.

On switching the indices on this last equation we have that

$$\langle\,\hbar\gamma_\mu\partial_\nu\psi i\gamma_3\widetilde{\psi}\,\rangle_1 = \rho\big(v_\mu P_\nu + (v\wedge\gamma_\mu)\cdot W_\nu\big)\,. \tag{161}$$

Next, remembering that we introduced $T_{\mu\nu}$ as

$$T_{\mu\nu} = \hbar\langle\,\gamma_\mu(\partial_\nu\psi)i\gamma_3\widetilde{\psi}\,\rangle - e\rho v_\mu A_\nu\,. \tag{162}$$

From these last two equation we get

$$T_{\mu\nu} = \rho\big(v_\mu P_\nu + (v\wedge\gamma_\mu)\cdot W_\nu\big) - e\rho v_\mu A_\nu\,. \tag{163}$$

Then, using Eq. (3.20):

$$P_\mu = p_\mu + eA_\mu\,, \tag{164}$$

and this equation and the one before it, we get

$$\begin{aligned}T_{\mu\nu} &= \rho\big(v_\mu[\,p_\mu + eA_\mu\,] + (v\wedge\gamma_\mu)\cdot W_\nu\big) - e\rho v_\mu A_\nu \\ &= \rho\big(v_\mu p_\mu + (v\wedge\gamma_\mu)\cdot W_\nu\big)\,. \end{aligned} \tag{165}$$

along with

$$T_\mu = \rho v_\mu p + N_\mu\,, \tag{166}$$

we get

$$T_{\mu\nu} = \rho v_\mu p_\nu + N_{\mu\nu}\,, \tag{167}$$

which is Eq. (3.21). So, on comparing (167) and (165) we have that

$$N_{\mu\nu} = \rho(v\wedge\gamma_\mu)\cdot W_\nu\,. \tag{168}$$

Thus, using that $(v \wedge \gamma_\mu) \cdot S = (v \wedge \gamma_\mu) \cdot (is \wedge v) \equiv 0$:

$$\begin{aligned} N_{\mu\nu} &= \rho(v \wedge \gamma_\mu) \cdot W_\nu \\ &= \rho(v \wedge \gamma_\mu) \cdot [\, \partial_\nu S + S(\partial_\nu \ln \rho + i\partial_\nu \beta)\,] \\ &= \rho(v \wedge \gamma_\mu) \cdot \partial_\nu S + \rho(v \wedge \gamma_\mu) \cdot [\, S(\partial_\nu \ln \rho + i\partial_\nu \beta)\,] \\ &= \rho(v \wedge \gamma_\mu) \cdot \partial_\nu S + \rho(v \wedge \gamma_\mu) \cdot (Si\partial_\nu \beta) \\ &= \rho(v \wedge \gamma_\mu) \cdot \partial_\nu S - \rho s_\mu \partial_\nu \beta \,, \end{aligned} \tag{169}$$

which is Eq. (3.22). Let's add some further calculations:

$$\begin{aligned} \rho(v \wedge \gamma_\mu) \cdot (Si\partial_\nu \beta) &= \rho(v \wedge \gamma_\mu) \cdot (vs\partial_\nu \beta) \\ &= \rho(\partial_\nu \beta)(-\gamma_\mu \cdot s) \\ &= -\rho s_\mu \partial_\nu \beta \,. \end{aligned} \tag{170}$$

4 Integrability Conditions

From (3.2) we have again

$$\Omega_\mu = 2(\partial_\mu R)\widetilde{R} \,, \tag{171}$$

which, by a little algebra, can be writtten in the form of Eq. (4.1):

$$\partial_\mu R = \tfrac{1}{2}\Omega_\mu R \,. \tag{172}$$

On differentiating this, we get (4.2):

$$\partial_\nu \partial_\mu R = \tfrac{1}{2}(\partial_\nu \Omega_\mu + \tfrac{1}{2}\Omega_\mu \Omega_\nu)R \,. \tag{173}$$

If we require the standard integrability condition on partial derivatives, then

$$\partial_\nu \partial_\mu R = \partial_\mu \partial_\nu R \,, \tag{174}$$

which is Eq. (4.3), then

$$\partial_\nu \Omega_\mu - \partial_\mu \Omega_\nu = \tfrac{1}{2}[\, \Omega_\nu \,, \Omega_\mu \,] \,, \tag{175}$$

which is Eq. (4.4).

To arrive at Eq. (4.5), we have a bit of work to do. So, let's begin with (128) [Eq. (3.6)]:

$$P_\mu + iq_\mu + \partial_\mu S = \Omega_\mu S \,, \tag{176}$$

and take its partial by ν:

$$\partial_\nu P_\mu + i\partial_\nu q_\mu + \partial_\nu \partial_\mu S = (\partial_\nu \Omega_\mu)S + \Omega_\mu \partial_\nu S \,. \tag{177}$$

On switching the indices, we get

$$\partial_\mu P_\nu + i\partial_\mu q_\nu + \partial_\mu \partial_\nu S = (\partial_\mu \Omega_\nu)S + \Omega_\nu \partial_\mu S \,. \tag{178}$$

Subtracting (177) from (178), we get

$$\partial_\mu P_\nu - \partial_\nu P_\mu + i(\partial_\mu q_\nu - \partial_\nu q_\mu) = (\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu)S + (\Omega_\nu \partial_\mu - \Omega_\mu \partial_\nu)S\,. \quad (179)$$

Using (4.3) we get

$$\partial_\mu P_\nu - \partial_\nu P_\mu + i(\partial_\mu q_\nu - \partial_\nu q_\mu) = \tfrac{1}{2}[\,\Omega_\mu\,, \Omega_\nu\,]S + (\Omega_\nu \partial_\mu - \Omega_\mu \partial_\nu)S\,. \quad (180)$$

Obviously, I need to do more work on it to get Eq. (4.5). The correct answer is given as

$$\partial_\mu P_\nu - \partial_\nu P_\mu + i(\partial_\mu q_\nu - \partial_\nu q_\mu) = \tfrac{1}{2}[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\,. \quad (181)$$

The scalar part of this last equation is Eq. (4.6):

$$\begin{aligned}\partial_\mu P_\nu - \partial_\nu P_\mu &= \tfrac{1}{2}[\,\partial_\nu S, \partial_\mu S\,] \cdot S^{-1} \\ &= \langle\, \partial_\nu S \wedge \partial_\mu S \cdot S^{-1}\,\rangle \\ &= \langle\, \partial_\nu S \partial_\mu S S^{-1}\,\rangle\,. \end{aligned} \quad (182)$$

Now, we make a substitution: $P_\mu = p_\mu + eA_\mu$:

$$\partial_\mu p_\nu - \partial_\nu p_\mu + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = (\partial_\nu S \partial_\mu S) \cdot S^{-1}\,, \quad (183)$$

and this corrects a misprint in the preprint version of this equation. But

$$\partial_\mu A_\nu - \partial_\nu A_\mu = (\gamma_\nu \wedge \gamma_\mu) \cdot (\Box \wedge A) = F_{\mu\nu}\,. \quad (184)$$

Therefore,

$$\partial_\mu p_\nu - \partial_\nu p_\mu + eF_{\mu\nu} = (\partial_\nu S \partial_\mu S) \cdot S^{-1}\,, \quad (185)$$

which is Eq. (4.7). However, my version of (4.7) comes out to be

$$\partial_\mu p_\nu - \partial_\nu p_\mu + eF_{\mu\nu} = (\partial_\nu S \wedge \partial_\mu S) \cdot S^{-1}\,. \quad (186)$$

The reason these two equations are equivalent is because $(\partial_\nu S \cdot \partial_\mu S) \cdot S^{-1} \equiv 0$, because the inner product of any multivector with a scalar is zero.

5 Physical Content of the Dirac Equation

We want the Dirac equation expressed in local variables to ferret out its physical content. By multiplying (55) on the right by $\widetilde{\psi}$, we get Eq. (5.1)

$$\begin{aligned}\hbar(\Box\psi)i\gamma_3\gamma_0\widetilde{\psi} &= m\psi\gamma_0\widetilde{\psi} + eA\psi\widetilde{\psi} \\ &= m\rho v + eA\rho e^{i\beta}\,. \end{aligned} \quad (187)$$

Using (156) and

$$P_\mu + iq_\mu + \partial_\mu S = \Omega_\mu S = \hbar \partial_\mu R \gamma_2 \gamma_1 \widetilde{R}\,, \quad (188)$$

we get Eq. (5.2):

$$\begin{aligned}\hbar(\partial_\mu\psi)\gamma_2\gamma_1\widetilde{\psi} &= [\,\partial_\mu(\rho e^{i\beta}) + \Omega_\mu\rho e^{i\beta}\,]\frac{\hbar}{2}R\gamma_2\gamma_1\widetilde{R}\\ &= \partial_\mu(\rho e^{i\beta})S + (P_\mu + iq_\mu + \partial_\mu S)\rho e^{i\beta}\\ &= (P_\mu + iq_\mu)\rho e^{i\beta} + \partial_\mu(\rho e^{i\beta}S)\,. \end{aligned}\tag{189}$$

But let's do this in parts. First, we remember that $\psi = \rho^{1/2}e^{\frac{1}{2}i\beta}R$. Thus,

$$\begin{aligned}\hbar(\partial_\mu\psi)\gamma_2\gamma_1\widetilde{\psi} &= \hbar[\,\tfrac{1}{2}\rho^{-1/2}(\partial_\mu\rho)e^{\frac{1}{2}i\beta}R + \tfrac{1}{2}\rho^{1/2}(i\partial_\mu\beta)e^{\frac{1}{2}i\beta}R\\ &\quad + \rho^{1/2}e^{\frac{1}{2}i\beta}(\partial_\mu R)\,]\,\rho^{1/2}e^{\frac{1}{2}i\beta}\gamma_2\gamma_1\widetilde{R}\\ &= \frac{\hbar}{2}[\,(\partial_\mu\rho)e^{i\beta} + \rho e^{i\beta}(\partial_\mu\beta) + \rho e^{i\beta}2(\partial_\mu R)\widetilde{R}\,]\,R\gamma_2\gamma_1\widetilde{R}\\ &= [\,\partial_\mu(\rho e^{i\beta}) + \Omega_\mu\rho e^{i\beta}\,]\frac{\hbar}{2}R\gamma_2\gamma_1\widetilde{R}\,. \end{aligned}\tag{190}$$

Next,

$$\begin{aligned}[\,\partial_\mu(\rho e^{i\beta}) + \Omega_\mu\rho e^{i\beta}\,]\frac{\hbar}{2}R\gamma_2\gamma_1\widetilde{R} &= [\,\partial_\mu(\rho e^{i\beta}) + \Omega_\mu\rho e^{i\beta}\,]S\\ &= \partial_\mu(\rho e^{i\beta})S + \Omega_\mu S\rho e^{i\beta}\\ &= \partial_\mu(\rho e^{i\beta})S + (P_\mu + iq_\mu + \partial_\mu S)\rho e^{i\beta}\\ &= (P_\mu + iq_\mu)\rho e^{i\beta} + \partial_\mu(\rho e^{i\beta}S)\,, \end{aligned}\tag{191}$$

where we used (176) and then recollected terms. Multiply through (190) on the left by γ^μ, we get

$$\hbar(\Box\psi)\gamma_2\gamma_1\widetilde{\psi} = (P + qi)\rho e^{i\beta} + \Box(\rho e^{i\beta}S)\,,\tag{192}$$

which is Eq. (5.3). Putting (187) and (192) together we have that

$$\rho e^{-i\beta}(p - iq) = \rho m v - \Box(\rho e^{i\beta}S)\,,\tag{193}$$

which is Eq. (5.4). Multiply this through by i on the right to get:

$$\rho e^{-i\beta}(-ip - q) = \rho m v i - \Box(\rho e^{i\beta}iS)\,,\tag{194}$$

which simplifies to

$$\rho e^{-i\beta}(ip + q) = \rho m i v + \Box(\rho e^{i\beta}iS)\,.\tag{195}$$

So, why did we multiply (193) by the unit pseudoscalar? We did because we want the pseudovector part of (193), and we can get at that by taking the vector part of (195), which gives us

$$\rho(p\sin\beta + q\cos\beta) = \Box\cdot(\rho e^{i\beta}iS)\,,\tag{196}$$

which is Eq. (5.5). The vector part of (193) gives

$$\rho(p\cos\beta - q\sin\beta) = \rho m v + \Box\cdot(\rho e^{i\beta}S)\,, \tag{197}$$

which is Eq. (5.6).

Moving on to Eq. (5.7). Let's return to (193) and distribute the derivative operator.

$$\begin{aligned}\rho e^{-i\beta}(p - iq) &= \rho m v - \gamma^\mu \partial_\mu(\rho e^{i\beta}S)\\ &= \rho m v - \gamma^\mu[\,(\partial_\mu\rho)e^{i\beta}S) + \rho(\partial_\mu e^{i\beta})S + \rho e^{i\beta}\partial_\mu S\,]\\ &= \rho m v - (\Box\rho)e^{i\beta}S - \rho(\Box e^{i\beta})S - \rho e^{-i\beta}\Box S\\ &= \rho m v - e^{-i\beta}(\Box\rho)S - \rho e^{-i\beta}(\Box i\beta)S - \rho e^{-i\beta}\Box S\\ &= \rho m v - e^{-i\beta}(\Box\rho)S + \rho i e^{-i\beta}(\Box\beta)S - \rho e^{-i\beta}\Box S\,.\end{aligned} \tag{198}$$

Therefore,

$$\begin{aligned}\rho(p - iq) &= \rho m e^{i\beta}v - (\Box\rho)S + \rho i(\Box\beta)S - \rho\Box S\\ &= \rho m e^{i\beta}v - \Box(\rho S) + i(\Box\beta)\rho S\,,\end{aligned} \tag{199}$$

which is Eq. (5.7). The vector part of this last equation gives us (5.8):

$$\rho p = \rho m v\cos\beta - \Box\cdot(\rho S) + \rho(iS)\cdot(\Box\beta)\,. \tag{200}$$

Note: The equation embedded in the text at the bottom of page 16 of the preprint contains an error. It should read

$$(iS)\cdot\Box\beta = (v\wedge s)\cdot\Box\beta = v\,s\cdot\Box\beta - s\,v\cdot\Box\beta\,. \tag{201}$$

Now, the trivector part of (199) can be determined by taking the vector part of its dual. But before I take its dual, I want to put it into a more convenient form, such as:

$$\rho(p - iq) = \rho m e^{i\beta}v + i\Box(\rho s v) + i(\Box\beta)\rho S\,. \tag{202}$$

On taking the dual of this last equation gives us

$$\rho(ip + q) = \rho m i e^{i\beta}v - \Box(\rho s v) - (\Box\beta)\rho S\,, \tag{203}$$

and taking the vector part of this gives us

$$-\rho q = \rho m v\sin\beta + \Box\cdot(\rho s v) + (\Box\beta)\cdot(\rho S)\,, \tag{204}$$

which is Eq. (5.9).

The paper claims that by use of (64) and (145), this last equation can be transformed into

$$S\cdot\Box\beta = s\cdot(\Box\wedge v) - v\cdot\Box s = s\cdot\Box v - v\cdot(\Box\wedge s)\,. \tag{205}$$

which is Eq. (5.10).

I will begin by using (64) to replace the $\sin\beta$ term in (204) and use (145) to replace the q in the same equation:

$$-\rho[\,-v\cdot(\Box\wedge s) + v\cdot\Box s\,] = -v\Box\cdot(\rho s) + \Box\cdot(\rho s v) + (\Box\beta)\cdot(\rho S)\,. \tag{206}$$

Now, for the first simplification.

$$v\cdot(\Box\wedge s) - v\cdot\Box s = \rho^{-1}[\,-v\Box\cdot(\rho s) + \Box\cdot(\rho s v)\,] + S\cdot\Box\beta\,. \tag{207}$$

Obviously, we have to get rid of ρ. So, let's expand the divergences.

$$\begin{aligned}\Box\cdot(\rho s) &= \gamma^\mu\partial_\mu\cdot(\rho s)\\ &= (\partial_\mu\rho)\gamma^\mu\cdot s + \rho\gamma^\mu\partial_\mu\cdot s\\ &= s\cdot\Box\rho + \rho\Box\cdot s\,.\end{aligned} \tag{208}$$

And,

$$\begin{aligned}\Box\cdot(\rho s v) &= \gamma^\mu\partial_\mu\cdot(\rho s\wedge v)\\ &= \partial_\mu\rho\,\gamma^\mu\cdot(s\wedge v)\\ &= (\partial_\mu\rho)\gamma^\mu\cdot(s\wedge v) + \rho\partial_\mu\gamma^\mu\cdot(s\wedge v)\\ &= (\partial_\mu\rho)(s^\mu v - v^\mu s) + \rho\partial_\mu(s^\mu v - v^\mu s)\\ &= vs\cdot\Box\rho - sv\cdot\Box\rho + \rho(v\Box\cdot s - s\Box\cdot v)\,.\end{aligned} \tag{209}$$

On taking the square-bracket expression of (207), we have that

$$\begin{aligned}-v\,\Box\cdot(\rho s) + \Box\cdot(\rho s v) &= -v\{s\cdot\Box\rho + \rho\Box\cdot s\}\\ &\qquad + \{vs\cdot\Box\rho - sv\cdot\Box\rho + \rho v\Box\cdot s - \rho s\Box\cdot v\}\\ &= -vs\cdot\Box\rho - \rho v\Box\cdot s\\ &\qquad + \{vs\cdot\Box\rho - sv\cdot\Box\rho + \rho v\Box\cdot s - \rho s\Box\cdot v\}\\ &= -sv\cdot\Box\rho - \rho s\Box\cdot v\\ &= -s[\,v\cdot\Box\rho + \rho\Box\cdot v\,]\\ &= -s\Box\cdot(\rho v) = 0\,.\end{aligned} \tag{210}$$

Anyway, substituting this back into (207), yields

$$v\cdot\Box s = \dot{s} = v\cdot(\Box\wedge s) + S\cdot\Box\beta\,.$$

But we also need the following result:

$$
\begin{aligned}
S \cdot \Box\beta &= \langle\, S\Box\beta \,\rangle_1 \\
&= \langle\, isv\Box\beta \,\rangle_1 \\
&= -\langle\, siv\Box\beta \,\rangle_1 \\
&= -s \cdot \langle\, iv\Box\beta \,\rangle_2 \\
&= -s \cdot (iv \wedge \Box\beta)\,. \qquad (211)
\end{aligned}
$$

Now, Eq. (5.11) of the preprint should be written as

$$
v \cdot \Box s = \dot{s} = s \cdot (\Box \wedge v) + s \cdot (iv \wedge \Box\beta)\,. \qquad (212)
$$

Proving this last equation from (205) is not too hard. We substitute out $S \cdot \Box\beta$ by use of the following identity:

$$
v \cdot \Box s \equiv (is \wedge v) \cdot \Box\beta = -s \cdot (iv \wedge \Box\beta)\,. \qquad (213)
$$

Now, we try to derive Eq. (5.12). We start with Eq. (5.8) and dot it with v:

$$
\begin{aligned}
p \cdot v &= m\cos\beta - \rho^{-1} v \cdot [\,\Box \cdot (\rho S)\,] + v \cdot [\,(iS) \cdot \Box\beta\,] \\
&= m\cos\beta - \rho^{-1} v \cdot [\,\Box \cdot (\rho isv)\,] + v \cdot [\,\Box\beta \cdot (iS)\,] \\
&= m\cos\beta - \rho^{-1} v \wedge \Box \cdot (\rho S) - (v \wedge \Box\beta) \cdot (iS)\,. \qquad (214)
\end{aligned}
$$

But,

$$
\begin{aligned}
\rho^{-1} v \cdot [\,\Box \cdot (\rho S)\,] &= \rho^{-1} v \cdot [\,\Box \cdot (\rho S)\,] \\
&= \rho^{-1} v \cdot (\Box\rho) \cdot S + \rho\Box \cdot S \\
&= \rho^{-1} (v \wedge \Box\rho) \cdot S + v \cdot \Box \cdot S \\
&= \rho^{-1} (v \wedge \Box\rho) \cdot S + v \wedge \Box \cdot S\,. \qquad (215)
\end{aligned}
$$

And,

$$
\begin{aligned}
(v \wedge \Box\beta) \cdot (iS) &= \langle\, (v \wedge \Box\beta)(iS) \,\rangle \\
&= \langle\, Si(v \wedge \Box\beta) \,\rangle \\
&= S \cdot (iv \wedge \Box\beta)\,. \qquad (216)
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
p \cdot v &= m\cos\beta - \rho^{-1} v \wedge \Box \cdot (\rho S) - (v \wedge \Box\beta) \cdot (iS) \\
&= m\cos\beta - \rho^{-1} (v \wedge \Box\rho) \cdot S + \Box \wedge v \cdot S + S \cdot (iv \wedge \Box\beta) \\
&= m\cos\beta - \rho^{-1} (v \wedge \Box\rho) \cdot S + S \cdot (\Box \wedge v + iv \wedge \Box\beta) \\
&= m\cos\beta - \rho^{-1} (v \wedge \Box\rho) \cdot S + S \cdot (\Box \wedge v + iv \wedge \Box\beta)\,. \qquad (217)
\end{aligned}
$$

Now, we try to derive Eq. (5.13). We start with Eq. (5.8) in the following form:

$$\begin{aligned} p &= mv\cos\beta - \rho^{-1}\Box\cdot(\rho S) + (iS)\cdot(\Box\beta) \\ &= mv\cos\beta - \rho^{-1}[\,(\Box\rho)\cdot S + \rho\Box\cdot S\,] + (v\wedge s)\cdot(\Box\beta) \\ &= mv\cos\beta - \rho^{-1}(\Box\rho)\cdot S - \Box\cdot S + (v\wedge s)\cdot(\Box\beta)\,. \end{aligned} \tag{218}$$

Rememebring that $v\cdot s = 0$, we dot this last equation by s^{-1},

$$\begin{aligned} p\cdot s^{-1} &= -s^{-1}\cdot\Box\cdot S + s^{-1}\cdot(v\wedge s)\cdot(\Box\beta) \\ &= -s^{-1}\cdot\Box\cdot S + (v\wedge s)\cdot(\Box\beta\wedge s^{-1}) \\ &= -s^{-1}\cdot\Box\cdot S + (v\wedge s)\cdot(\Box\beta\wedge s^{-1}) \\ &= -s^{-1}\cdot\Box\cdot S - v\cdot\Box\beta\,. \end{aligned} \tag{219}$$

Therefore

$$\dot\beta = v\cdot\Box\beta = -p\cdot s^{-1} - s^{-1}\cdot\Box\cdot S\,, \tag{220}$$

which is near to Eq. (5.13).

6 Proper Flows

Here, we are interested in the flow of local variables along a streamline. We begin with the proper angular velocity Ω along streamlines.

$$\Omega = 2\dot R\widetilde{R} = 2(v\cdot\Box R)\widetilde{R} = v^\mu\Omega_\mu\,. \tag{221}$$

First, an identity,

$$\Omega = 2\dot R\widetilde{R} = \{(\Box R)\widetilde{R}, v\} - \Box v\,, \tag{222}$$

which is Eq. (6.2). To establish this, we will need our old friend

$$ba = 2a\cdot b - ab\,. \tag{223}$$

for vectors a, b. Then

$$\begin{aligned} \Box vR &= \gamma^\mu\partial_\mu vR \\ &= \gamma^\mu(\partial_\mu v)R + \gamma^\mu v\partial_\mu R \\ &= (\Box v)R + \gamma^\mu v\partial_\mu R \\ &= (\Box v)R + (2\gamma^\mu\cdot v\partial_\mu - v\gamma^\mu\partial_\mu)R \\ &= (\Box v)R + 2v\cdot\Box R - v\Box R\,. \end{aligned} \tag{224}$$

And one more identity:

$$(\Box R)\widetilde{R}vR = \Box R\gamma_0 = \Box(vR)\,. \tag{225}$$

Hence,

$$(\Box R)\widetilde{R}vR = (\Box v)R + 2v\cdot\Box R - v\Box R\,. \tag{226}$$

$$(\Box R)\widetilde{R}v = \Box v + 2(v\cdot\Box R)\widetilde{R} - (v\Box R)\widetilde{R}$$
$$= \Box v + 2\dot{R}\widetilde{R} - (v\Box R)\widetilde{R}\,, \tag{227}$$

which gives us

$$2\dot{R}\widetilde{R} = (\Box R)\widetilde{R}v + v(\Box R)\widetilde{R} - \Box v$$
$$= \{(\Box R)\widetilde{R}, v\} - \Box v\,. \tag{228}$$

But to make progress, we need more than vector identities. So, we write the Dirac equation (2.15) in the form as

$$\hbar\Box\psi i\gamma_3\gamma_0\widetilde{\psi} = m\psi\gamma_0\widetilde{\psi} + eA\psi\widetilde{\psi}$$
$$= m\rho v + eA\rho e^{i\beta}\,. \tag{229}$$

But we can also write the Dirac Equation as

$$\hbar\Box\psi i\gamma_3\gamma_0\widetilde{\psi} = 2(\Box\psi)\widetilde{\psi}S = m\rho v + eA\rho e^{i\beta}\,. \tag{230}$$

And still more identities to deal with:

$$2(\Box\psi)\widetilde{\psi} = 2(\Box\rho^{1/2}e^{i\beta}R)\rho^{1/2}e^{\frac{1}{2}i\beta}\widetilde{R}$$
$$= [\Box\ln\rho + (\Box\beta)i + (\Box R)\widetilde{R}]\rho e^{i\beta}\,, \tag{231}$$

which is Eq. (6.4). From this we get,

$$(m\rho v + eA\rho e^{i\beta})S^{-1} = [\Box\ln\rho - i\Box\beta + 2(\Box R)\widetilde{R}]\rho e^{i\beta}\,, \tag{232}$$

which simplifies to

$$(mve^{-i\beta} + eA)S^{-1} = \Box\ln\rho - i\Box\beta + 2(\Box R)\widetilde{R}\,. \tag{233}$$

Thus, we get that

$$2(\Box R)\widetilde{R} = (mve^{-i\beta} + eA)S^{-1} - \Box\ln\rho + i\Box\beta\,, \tag{234}$$

which compares to Eq. (6.5) of the preprint.

Now, for our next trick, we need to understand that for trivector B and for vector v

$$B\cdot v = \tfrac{1}{2}(Bv + vB) = \tfrac{1}{2}\{B, v\}\,, \tag{235}$$

Therefore, for (6.6),

$$\tfrac{1}{2}\{2(\Box R)\widetilde{R}, v\} = [\,2(\Box R)\widetilde{R}\,]\cdot v$$
$$= [\,(mve^{-i\beta} + eA)S^{-1} - \Box\ln\rho + i\Box\beta\,]\cdot v$$
$$= -v\cdot\Box\ln\rho + v\cdot(i\Box\beta) + v\cdot[\,(mve^{-i\beta} + eA)S^{-1}\,]$$
$$= -v\cdot\Box\ln\rho + v\cdot(i\Box\beta) + v\cdot(mve^{-i\beta} + eA)S^{-1}\,, \tag{236}$$

since $v \cdot S^{-1} = 0$.

But how did we move the v on the right side to the left side? Sample calculation:

$$\begin{aligned}(i\Box\beta) \cdot v &= \langle\, (i\Box\beta)v \,\rangle_2 \\ &= -\langle\, (i\Box\beta)v \,\rangle_2^{\sim} \\ &= -\langle\, v(\Box\beta)i \,\rangle_2 \\ &= \langle\, vi(\Box\beta) \,\rangle_2 \\ &= v \cdot (i\Box\beta)\,. \end{aligned} \tag{237}$$

So, using (236) into (222), yields Eq. (6.7):

$$\Omega = -\Box \wedge v + v \cdot (i\Box\beta) + v \cdot (mv\cos\beta + eA)S^{-1}\,. \tag{238}$$

Hint: After substitution, keep only the bivector parts, as Ω is a bivector.

And from this, we get Eqs. (6.8)–(6.10):

$$\dot{v} = \Omega \cdot v = v \cdot (\Box \wedge v)\,, \tag{239a}$$

$$\dot{s} = \Omega \cdot s = s \cdot (\Box \wedge v) + s \cdot [\, v \cdot (\Box\beta i)\,] \tag{239b}$$

$$\dot{S} = \tfrac{1}{2}[\,\Omega, S\,] = \tfrac{1}{2}[\,S, \Box \wedge v\,] + \tfrac{1}{2}[\,S, v \cdot (\Box\beta i)\,]\,, \tag{239c}$$

Equation (182) can be re-expressed as

$$\Box \wedge P = 0\,, \tag{240}$$

which is Eq. (6.11). Hence, P must be the gradient of some function, say χ:

$$P = \Box\chi\,, \tag{241}$$

where χ is the phase of the Dirac wave function. Going over to the classical limit, $\beta = 0$, $\sin\beta = 0$, and $\cos\beta = \pm 1$. Then, (199) becomes by taking the vector part:

$$p = \pm mv\,, \tag{242}$$

where we have set $\Box\beta = 0$, and set

$$\langle\, \Box(\rho S)\,\rangle_1 = \Box \cdot (\rho S) = 0\,. \tag{243}$$

Now, from (164) we can write

$$P = p + eA\,, \tag{244}$$

and then

$$p = \Box\chi - eA\,. \tag{245}$$

Therefore (242) becomes

$$p = \pm mv = \Box\chi - eA\,, \tag{246}$$

which is (6.13), where the $+,-$ signs refer to different charges. Squaring this gives the Hamilton-Jacobi equation (6.14):

$$(\Box\chi - eA)^2 = m^2 \,, \tag{247}$$

since $v^2 = 1$. Anyway, χ can be solved for if A is known.

On taking the curl of (246), we have that

$$\pm m\Box \wedge v = \Box \wedge \Box\chi - e\Box \wedge A = -e\Box \wedge A \,, \tag{248}$$

since $\Box \wedge \Box = 0$ as an identity. Continuing, we also know that $\Box \wedge A = F$, the electromagnetic tensor. Thus, the proper angular velocity is

$$\Omega = -\Box \wedge v = \pm\frac{e}{m}\, F \,, \tag{249}$$

which is Eq. (6.16) and which obtains the Lorentz force. Or, more generally,

$$\dot{R} = \pm\frac{e}{2m}\, FR \,, \tag{250}$$

which is Eq. (6.17).

Now for the dervation. From (222) we get

$$\dot{R} = \tfrac{1}{2}\Omega R \,. \tag{251}$$

Now we just substitute in from (249).

We next take (5.7):

$$\rho(p - iq) = \rho m v e^{i\beta} - \Box(\rho S) + i(\Box\beta)\rho S \,, \tag{252}$$

and put it in the form (6.18):

$$(P + iq) - eA = m v e^{-i\beta} - \gamma^\nu W_\nu \,, \tag{253}$$

and W_ν is given by (158).

So, we begin with (252) and use that $p = P - eA$, to get

$$\rho[\,(P + qi) - eA\,] = \rho m v e^{i\beta} - \Box\,(\rho S) + i(\Box\beta)\rho S \,. \tag{254}$$

Then, divide through by ρ:

$$(P + qi) - eA = m v e^{i\beta} - \rho^{-1}\Box\,(\rho S) + i(\Box\beta) S \,. \tag{255}$$

But

$$\begin{aligned} \Box(\rho S) &= \gamma^\nu \partial_\nu(\rho S) \\ &= \gamma^\nu[\,(\partial_\nu\rho) S) + \rho\partial_\nu S\,] \\ &= (\Box\rho) S + \rho\Box S \,. \end{aligned} \tag{256}$$

Hence,

$$\begin{aligned}\rho^{-1}\Box(\rho S) &= \rho^{-1}(\Box\rho)S + \Box S \\ &= (\Box \ln\rho)S + \Box S\,. \end{aligned} \tag{257}$$

So, (255) becomes

$$(P + qi) - eA = mve^{i\beta} - (\Box \ln\rho)S - \Box S + i(\Box\beta)S\,. \tag{258}$$

Now,

$$\begin{aligned}\gamma^\nu W_\nu &= \gamma^\nu[\,\partial_\nu S + S(\partial_\nu \ln\rho + i\partial_\nu\beta)\,] \\ &= -\Box S - \gamma^\nu S(\partial_\nu \ln\rho + i\partial_\nu\beta) \\ &= -\Box S - \gamma^\nu(\partial_\nu \ln\rho + i\partial_\nu\beta)S \\ &= -\Box S - (\Box \ln\rho)S + i\Box\beta S\,. \end{aligned} \tag{259}$$

Finally,

$$(P + qi) - eA = mve^{i\beta} - \gamma^\nu W_\nu\,. \tag{260}$$

First, we take the gradient of this last equation, to get

$$(\Box P + \Box qi) - e\Box A = m[\,\Box v - i(\Box\beta)v\,]e^{-i\beta} + \gamma^\mu\gamma^\nu\partial_\mu W_\nu\,, \tag{261}$$

which is (6.19). Then, on taking the bivector part, we get

$$\begin{aligned} m[\,\Box\wedge v - i(\Box\beta)\wedge v\,]e^{-i\beta} &= -eF + \partial_\mu W^\mu + \tfrac{1}{2}[\,\gamma^\mu\wedge\gamma^\nu, \partial_\mu W_\nu\,] \\ &\quad + (\Box\wedge P + \Box\wedge q\,i)\,, \end{aligned} \tag{262}$$

which is (6.20).

As a partial calculation, let's find the bivector part of $\gamma^\mu\gamma^\nu\partial_\mu W_\nu$. For starters,

$$\gamma^\mu\gamma^\nu\partial_\mu W_\nu = \gamma^\mu\cdot\gamma^\nu\partial_\mu W_\nu + \gamma^\mu\wedge\gamma^\nu\partial_\mu W_\nu\,. \tag{263}$$

Then

$$\begin{aligned}\langle\,\gamma^\mu\gamma^\nu\partial_\mu W_\nu\,\rangle_2 &= \eta^{\mu\nu}\partial_\mu W_\nu + \langle\,\gamma^\mu\wedge\gamma^\nu\partial_\mu W_\nu\,\rangle_2 \\ &= \partial_\mu W^\mu + \tfrac{1}{2}[\,\gamma^\mu\wedge\gamma^\nu, \partial_\mu W_\nu\,]\,. \end{aligned} \tag{264}$$

Let's recast (181) into the form of (6.21):

$$\Box\wedge P + \Box\wedge q\,i = \tfrac{1}{4}\gamma^\mu\gamma^\nu[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\,, \tag{265}$$

from whence we from (262) that

$$\begin{aligned} m[\,\Box\wedge v - i(\Box\beta)\wedge v\,]e^{-i\beta} &= -eF + \partial_\mu W^\mu + \tfrac{1}{2}[\,\gamma^\mu\wedge\gamma^\nu, \partial_\mu W_\nu\,] \\ &\quad + \tfrac{1}{4}\gamma^\mu\gamma^\nu[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\,, \end{aligned} \tag{266}$$

And some more massaging:

$$\begin{aligned}\Box\wedge v - i(\Box\beta)\wedge v &= -\frac{eF}{m}e^{i\beta} + m^{-1}\{\partial_\mu W^\mu + \tfrac{1}{2}[\,\gamma^\mu\wedge\gamma^\nu, \partial_\mu W_\nu\,] \\ &\quad + \tfrac{1}{4}\gamma^\mu\gamma^\nu[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\}e^{i\beta} \\ &= -\frac{eF}{m}e^{i\beta} + m^{-1}e^{i\beta}\{\partial_\mu W^\mu + \tfrac{1}{2}[\,\gamma^\mu\wedge\gamma^\nu, \partial_\mu W_\nu\,] \\ &\quad + \tfrac{1}{4}\gamma^\mu\gamma^\nu[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\}\,. \end{aligned} \tag{267}$$

---

get that

$$\partial_\mu W_\nu - \partial_\nu W_\mu + \tfrac{1}{2}[\,\partial_\nu S, \partial_\mu S\,]S^{-1} = \tfrac{1}{2}[\,W_\mu, W_\nu\,]\,S^{-1}\,, \tag{268}$$

which is (6.22). (Note: On the RHS of (265), I get a factor of a half rather than of a fourth.)

Let's try this.

$$\partial_\mu W_\nu - \partial_\nu W_\mu = (\partial_\mu S)(W_\nu - \partial_\nu S)S^{-1} - (\partial_\nu S)(W_\mu - \partial_\mu S)S^{-1}\,. \tag{269}$$

Hence,

$$\begin{aligned}\partial_\mu W_\nu - \partial_\nu W_\mu - [\,(\partial_\mu S)W_\nu - (\partial_\nu S)\,W_\mu\,]\,S^{-1} &= -[\,(\partial_\mu S)(\partial_\nu S) - (\partial_\nu S)(\partial_\mu S)\,]S^{-1} \\ &= [\,\partial_\nu S, \partial_\mu S\,]S^{-1}\,. \end{aligned} \tag{270}$$

This can be easily rewritten as (268).

Combining these last results (262) finally becomes

$$\Box\wedge v + iv\wedge\Box\beta = -\frac{e}{m}Fe^{i\beta} + C\,, \tag{271}$$

where

$$mC = e^{i\beta}\big(\partial_\mu W^\mu + \tfrac{1}{4}[\,\gamma^\mu\gamma^\nu, [\,W_\mu, W_\nu\,]S^{-1}\,]\big)\,. \tag{272}$$

Now for a short lemma.

$$\begin{aligned} iv\wedge\Box\beta &= \langle\, iv\wedge\Box\beta\,\rangle_2 \\ &= \langle\, iv\Box\beta\,\rangle_2 \\ &= -\langle\, vi\Box\beta\,\rangle_2 \\ &= -v\cdot(i\Box\beta)\,. \end{aligned} \tag{273}$$

Next, we shoot for (6.25). Substituting (271) into (238), we get

$$\begin{aligned}\Omega &= -\Box\wedge v + v\cdot(i\Box\beta) + v\cdot(mv\cos\beta + eA)S^{-1} \\ &= [\,iv\wedge\Box\beta + \frac{e}{m}Fe^{i\beta} - C\,] + v\cdot(i\Box\beta) + v\cdot(mv\cos\beta + eA)S^{-1} \\ &= \frac{e}{m}Fe^{i\beta} - C + v\cdot(mv\cos\beta + eA)S^{-1}\,, \end{aligned} \tag{274}$$

which is Eq. (6.25).

Eq. (6.26) follows immediately from $\dot{v} = \Omega \cdot v$:

$$\dot{v} = \frac{e}{m}(Fe^{i\beta}) \cdot v + v \cdot C \,. \tag{275}$$

With a little more effort, we get Eq. (6.27):

$$\dot{S} = \tfrac{1}{2}\big[\, F, \frac{e}{m} S e^{i\beta} \,\big] + \tfrac{1}{2}[\, S, C\,] \,. \tag{276}$$

To prove this, we begin with $2\dot{S} = [\,\Omega, S\,]$.

$$\begin{aligned}
[\,\Omega, S\,] &= [\,\frac{e}{m}Fe^{i\beta} - C + v \cdot (mv\cos\beta + eA)S^{-1}, S\,] \\
&= \frac{e}{m}FSe^{i\beta} - CS + v \cdot (mv\cos\beta + eA) \\
&\quad - \{\frac{e}{m}SFe^{i\beta} - SC + v \cdot (mv\cos\beta + eA)\} \\
&= \frac{e}{m}e^{i\beta}[\,F, S\,] - [\,C, S\,] \\
&= [\,F, \frac{e}{m}Se^{i\beta}\,] + [\,S, C\,] \,,
\end{aligned} \tag{277}$$

Hence,

$$\dot{S} = \tfrac{1}{2}\big[\, F, \frac{e}{m} S e^{i\beta} \,\big] + \tfrac{1}{2}[\, S, C\,] \,. \tag{278}$$

We now introduce the local magnetic moment suggests that

$$\mu = \frac{|\,e\,|}{m} S e^{i\beta} \,. \tag{279}$$

and

$$|\,\mu\,| = \frac{|\,e\,|}{m}|\,S\,| = \frac{|\,e\,|\hbar}{2m} \,. \tag{280}$$

From (274) and (128) we get

$$v \cdot (p + eA) + iv \cdot q + \dot{S} = \Omega S = \frac{e}{m}Fe^{i\beta}S - CS + m\cos\beta + ev \cdot A \,. \tag{281}$$

After cancelling the $ev \cdot A$ term from both sides, we get

$$v \cdot p + iv \cdot q + \dot{S} = \Omega S = \frac{e}{m}Fe^{i\beta}S - CS + m\cos\beta \,. \tag{282}$$

The pseudoscalar part of (282) gives

$$iv \cdot q = \frac{e}{m}\langle\, Fe^{i\beta}S\,\rangle_4 - C \wedge S \,. \tag{283}$$

Hence, we get (6.31)

$$
\begin{aligned}
v\cdot q &= (-i)\big[\,\frac{e}{m}\langle\, Fe^{i\beta}S\,\rangle_4 - C\wedge S\,\big] \\
&= i\big[\,C\wedge S - \frac{e}{m}\big(Fe^{i\beta}\big)\wedge S\,\big] \\
&= \frac{e}{m}(Fe^{i\beta})\cdot(s\wedge v) + C\cdot(v\wedge s)\,, \qquad (284)
\end{aligned}
$$

where we used that $S = isv = is\wedge v$. From the scalar part of (282) is (6.32)

$$
p\cdot v = m\cos\beta + \Big(\frac{e}{m}Se^{i\beta}\Big)\cdot F - C\cdot S\,. \qquad (285)
$$

$$
\begin{aligned}
p\cdot v &= \frac{e}{m}\langle\, Fe^{i\beta}S\,\rangle - C\cdot S + m\cos\beta \\
&= \frac{e}{m}\langle\, Se^{i\beta}F\,\rangle - C\cdot S + m\cos\beta \\
&= \langle\,\frac{e}{m}e^{i\beta}S\,\rangle_2\cdot F - C\cdot S + m\cos\beta \\
&= \big(\frac{e}{m}e^{i\beta}S\big)\cdot F - C\cdot S + m\cos\beta\,. \qquad (286)
\end{aligned}
$$

Things to know to produce Eq. (6.33): Start with (168),

$$
\begin{aligned}
N_{\mu\nu} = N_\mu\cdot\gamma_\nu &= \rho(v\wedge\gamma_\mu)\cdot W_\nu \\
&= \rho(v\wedge\gamma_\mu)\cdot\partial_\nu S - \rho s_\mu\partial_\nu\beta\,. \qquad (3.22)
\end{aligned}
$$

$$
\Box\cdot(\rho s) = -m\rho\sin\beta\,. \qquad (2.18)
$$

$$
\Box\cdot(\rho s) = s\cdot\Box\rho + \rho\Box\cdot s\,. \qquad (287)
$$

$$
\Box\wedge v + iv\wedge\Box\beta = -\frac{e}{m}Fe^{i\beta} + C\,. \qquad (6.23)
$$

It will a bit of effort to prove Eq. (3.22). We'll start with (158):

$$
W_\mu = (\rho e^{i\beta})^{-1}\partial_\nu(\rho e^{i\beta}S)\,. \qquad (288)
$$

So,

$$
\begin{aligned}
\rho(v\wedge\gamma_\mu)\cdot W_\nu &= \rho(v\wedge\gamma_\mu)\cdot[\,(\rho e^{i\beta})^{-1}\partial_\mu(\rho e^{i\beta}S)\,] \\
&= \rho(v\wedge\gamma_\mu)\cdot[\,(\rho e^{i\beta})^{-1}\{(\partial_\nu\rho)e^{i\beta}S) + \rho\partial_\nu(e^{i\beta})S\} + \partial_\nu S\,] \\
&= \rho(v\wedge\gamma_\mu)\cdot\partial_\nu S + \rho(v\wedge\gamma_\mu)\cdot[\,(\rho e^{i\beta})^{-1}\{(\partial_\nu\rho)e^{i\beta}S) + \rho\partial_\nu(e^{i\beta})S\}\,] \\
&= \rho(v\wedge\gamma_\mu)\cdot\partial_\nu S + \rho(v\wedge\gamma_\mu)\cdot[\,\rho^{-1}(\partial_\nu\rho)S + (\partial_\nu\beta)iS\,] \\
&= \rho(v\wedge\gamma_\mu)\cdot\partial_\nu S + \rho(v\wedge\gamma_\mu)\cdot[\,\rho^{-1}(\partial_\nu\rho)S - (\partial_\nu\beta)sv\,] \\
&= \rho(v\wedge\gamma_\mu)\cdot\partial_\nu S - \rho(v\wedge\gamma_\mu)\cdot[\,(\partial_\nu\beta)s\wedge v\,]\,, \qquad (289)
\end{aligned}
$$

where we used that $(v \wedge \gamma_\mu) \cdot \rho^{-1}(\partial_\nu \rho) S = 0$ because $v \cdot S = 0$. Continuing, we have that

$$\rho(v \wedge \gamma_\mu) \cdot W_\nu = \rho(v \wedge \gamma_\mu) \cdot \partial_\nu S - \rho s_\mu \partial_\nu \beta \,. \tag{290}$$

Next, we have one of the harder equations to establish, Eq. (6.33):

$$\begin{aligned}
\partial^\mu N_{\mu\nu} &= \rho S \cdot (\Box \wedge \partial_\nu v) - (\partial_\nu v \wedge \Box) \cdot (\rho S) - \rho s \cdot \Box\, \partial_\nu \beta - (\partial_\nu \beta) \Box \cdot \rho s \\
&= -\rho S \cdot \partial_\nu \left(\frac{e}{m} F e^{i\beta} - C \right) + \rho(\partial_\nu v \wedge \Box\, \beta) \cdot (iS) \\
&\qquad - (\partial_\nu v \wedge \Box) \cdot (\rho S) - m \partial_\nu \cos\beta \\
&= -\rho \Big(S \cdot \partial_\nu \frac{e}{m} (F e^{i\beta} - C) + (\partial_\nu v \wedge \gamma^\mu) \cdot W_\mu + m \partial_\nu \cos\beta \Big) \,.
\end{aligned} \tag{291}$$

Proof following.

Lemma: $v \wedge \gamma_\mu \cdot S = 0$. This uses the fact that $v \cdot S = 0$.

Definition: Given a differentiable operator D and differentiable functions F and G, then we are said to "contraflux" the derivatives of F and G, if we replace the expression $(DF)G$ by $D(FG) - F(DG)$.

Let's see how far we can get on this one. To begin with, we have that

$$\partial^\mu N_{\mu\nu} = \partial^\mu [\, \rho(v \wedge \gamma_\mu) \cdot \partial_\nu S - \rho s_\mu \partial_\nu \beta \,] \,. \tag{292}$$

Let's differentiate the terms separately. Careful observation of the first two terms on the RHS of the first line of (291) show that we need to bring the ρ and the S together. Sounds like a plan! It looks like we should begin by contrafluxing the derivative ∂_ν over the dot product of the two bivectors. Thus,

$$\begin{aligned}
\partial^\mu [\, \rho(v \wedge \gamma_\mu) \cdot \partial_\nu S \,] &= \partial^\mu [\, \rho \partial_\nu \{ (v \wedge \gamma_\mu) \cdot S \} - \rho \{ \partial_\nu (v \wedge \gamma_\mu) \} \cdot S \,] \\
&= -\partial^\mu [\, \{ \partial_\nu (v \wedge \gamma_\mu) \} \cdot (\rho S) \,] \\
&= -\{ \partial_\nu ((\partial^\mu v) \wedge \gamma_\mu) \} \cdot (\rho S) \; - \{ \partial_\nu (v \wedge \gamma_\mu \partial^\mu) \} \cdot (\rho S) \\
&= -(\rho S) \cdot \{ \partial_\nu ((\partial^\mu v) \wedge \gamma_\mu) \} \; - \{ \partial_\nu (v \wedge \Box) \} \cdot (\rho S) \\
&= (\rho S) \cdot \{ \partial_\nu (\Box \wedge v) \} \; - (\partial_\nu v \wedge \Box) \cdot (\rho S) \\
&= \rho S \cdot (\Box \wedge \partial_\nu v) - (\partial_\nu v \wedge \Box) \cdot (\rho S) \,.
\end{aligned} \tag{293}$$

And that gives us the first two terms of the first line of (291). Then,

$$\begin{aligned}
\partial^\mu [\, -\rho s_\mu \partial_\nu \beta \,] &= -(\partial^\mu \rho) s_\mu \partial_\nu \beta - \rho \partial^\mu [\, s_\mu \partial_\nu \beta \,] \\
&= -(s \cdot \Box \rho) \partial_\nu \beta - \rho [\, (\Box \cdot s) \partial_\nu \beta + s \cdot \Box \partial_\nu \beta \,] \\
&= -(\partial_\nu \beta) [\, (s \cdot \Box\, \rho) + \rho(\Box \cdot s) \,] - \rho s \cdot \Box \partial_\nu \beta \\
&= -(\partial_\nu \beta)\, \Box \cdot (\rho s) - \rho s \cdot \Box \partial_\nu \beta \,.
\end{aligned} \tag{294}$$

And that gives us the third and fourth terms of the first line of (291).

Now that we have

$$\partial^\mu N_{\mu\nu} = \rho S \cdot (\Box \wedge \partial_\nu v) - (\partial_\nu v \wedge \Box) \cdot (\rho S) - \rho s \cdot \Box\, \partial_\nu \beta - (\partial_\nu \beta)\Box \cdot \rho s\,, \quad (295)$$

can we get the next line of (291)? That is, (6.33)

$$\begin{aligned} \partial^\mu N_{\mu\nu} &= \rho S \cdot (\Box \wedge \partial_\nu v) - (\partial_\nu v \wedge \Box) \cdot (\rho S) - \rho s \cdot \Box\, \partial_\nu \beta - (\partial_\nu \beta)\Box \cdot \rho s \\ &= -\rho S \cdot \partial_\nu \left(\frac{e}{m} F e^{i\beta} - C \right) + \rho (\partial_\nu v \wedge \Box\, \beta) \cdot (iS) \\ &\qquad - (\partial_\nu v \wedge \Box) \cdot (\rho S) - m \partial_\nu \cos\, \beta\,. \end{aligned} \quad (296)$$

Let's begin with the first term and use (271) in the form

$$\Box \wedge v = -\frac{e}{m} F e^{i\beta} + C - i v \wedge \Box \beta\,, \quad (297)$$

So,

$$\begin{aligned} \rho S \cdot (\Box \wedge \partial_\nu v) &= \rho S \cdot \partial_\nu (\Box \wedge v) \\ &= \rho S \cdot \partial_\nu \left(-\frac{e}{m} F e^{i\beta} + C - i v \wedge \Box \beta \right) \\ &= \rho S \cdot \partial_\nu \left(-\frac{e}{m} F e^{i\beta} + C \right) - \rho S \cdot \partial_\nu \left(i v \wedge \Box \beta \right) \\ &= -\rho S \cdot \partial_\nu \left(\frac{e}{m} F e^{i\beta} - C \right) - \rho (\partial_\nu v \wedge \Box\, \beta) \cdot (iS)\,. \end{aligned} \quad (298)$$

However, the text show a plus sign in front of the second term. Anyway, we've one more step:

$$\begin{aligned} S \cdot \partial_\nu \left(i v \wedge \Box \beta \right) &= \langle\, S \partial_\nu \left(i v \wedge \Box \beta \right) \rangle \\ &= \langle\, (iS) \partial_\nu \left(v \wedge \Box \beta \right) \rangle \\ &= \langle\, (\partial_\nu v \wedge \Box \beta) \, (iS) \,\rangle \\ &= (\partial_\nu v \wedge \Box\, \beta) \cdot (iS)\,. \end{aligned} \quad (299)$$

The last thing I have to show is that

$$-\rho s \cdot \Box\, \partial_\nu \beta - (\partial_\nu \beta)\Box \cdot \rho s = -m \partial_\nu \cos\, \beta\,. \quad (300)$$

Using (64), I can show that

$$\begin{aligned} -(\partial_\nu \beta)\Box \cdot \rho s &= (\partial_\nu \beta) m \rho \sin \beta \\ &= -m \rho\, \partial_\nu \cos\, \beta\,. \end{aligned} \quad (301)$$

Finish this!!

I repeat Eq. (2.26)

$$\begin{aligned}\partial_\mu T^\mu &= \partial_\mu(\rho v^\mu p) + \partial_\mu N^\mu \\ &= \partial_\mu(\rho v^\mu)p + \rho v^\mu \partial_\mu p + \partial_\mu N^\mu \\ &= \rho v \cdot \Box\, p + \partial_\mu N^\mu \\ &= \rho \dot{p} + \partial_\mu N^\mu \\ &= \rho e F \cdot v = \rho f \,. \end{aligned} \tag{302}$$

Now we go to Eq. (6.34). We begin with Eq. (2.26):

$$\partial_\mu T^\mu = \rho \dot{p} + \partial_\mu N^\mu = \rho e F \cdot v = \rho f \,, \tag{303}$$

from which we get that

$$\dot{p} = eF \cdot v - \rho^{-1} \partial_\mu N^\mu \,. \tag{304}$$

To proceed, I make that assumption that

$$\partial_\mu N^\mu = \partial^\mu N_\mu = \partial^\mu \gamma^\nu N_{\mu\nu} = \gamma^\nu \partial^\mu N_{\mu\nu} \,. \tag{305}$$

Now we go to Eq. (6.35).

$$\begin{aligned}\partial_\mu M^\mu &= \rho \tfrac{1}{2}[\, S, \Box \wedge v\,] + \tfrac{1}{2}[\, v \wedge \Box, \rho S\,] \\ &= \rho \tfrac{1}{2}\left[F, \frac{e}{m} S e^{i\beta} \right] + \rho \tfrac{1}{2}[\, S, C\,] + \tfrac{1}{2}[\, v \wedge \gamma^\mu, W_\mu\,] \,. \end{aligned} \tag{6.35}$$

We begin with Eq. (2.34)

$$\rho \dot{S} + \rho\, p \wedge v = \gamma_\mu \wedge N^\mu - \partial_\mu M^\mu \,, \tag{2.34}$$

and apply (271):

$$\Box \wedge v + i v \wedge \Box \beta = -\frac{e}{m} F e^{i\beta} + C \,. \tag{306}$$

From (2.10)

$$T_\mu = \rho v_\mu p + N_\mu \,, \tag{307}$$

we get

$$T^\mu = \rho\, v^\mu p + N^\mu \,, \tag{308}$$

Then,

$$\gamma_\mu \wedge T^\mu = \rho\, v \wedge p + \gamma_\mu \wedge N^\mu \,. \tag{309}$$

Eq. (2.20) is

$$\gamma^\mu \wedge \bar{T}_\mu = T_\mu \wedge \gamma^\mu = i(\Box \wedge \rho s) = -\Box \cdot (i\rho s) \,. \tag{310}$$

Then, using (2.34) from a half-dozen equations ago:

$$
\begin{aligned}
\partial_\mu M^\mu &= -\rho\dot{S} + \gamma_\mu\wedge N^\mu + \rho v\wedge p \\
&= -\rho\dot{S} + \gamma_\mu\wedge T^\mu \\
&= \Box\cdot(i\rho s) - \rho\dot{S} \\
&= \Box\cdot(i\rho s) + \tfrac{1}{2}\rho[\,\Box\wedge v, S\,] + \tfrac{1}{2}[\,v\cdot(\Box\beta i), \rho S\,]\,,
\end{aligned}
\tag{311}
$$

where we used (239c). Now,

$$
\begin{aligned}
[\,v\wedge\Box, \rho S\,] &= v\wedge\Box(\rho S) - \dot{\overline{(\rho S)}}v\wedge\dot{\Box} \\
v\Box(\rho S) &= v\cdot\Box(\rho S) + v\wedge\Box(\rho S) \\
&= (v\cdot\Box\rho)S + \rho\dot{S} + v\wedge\Box(\rho S)\,.
\end{aligned}
\tag{312}
$$

$$
\langle\, v\Box(\rho S)\,\rangle_2 = v\wedge(\Box\cdot(\rho S))\,.
\tag{313}
$$

$$
\begin{aligned}
(\partial_\nu v\wedge\Box)\cdot(\rho S) &= [\,(\partial_\nu v)\wedge\Box\,]\cdot(\rho S) \\
&= \langle\,(\partial_\nu v)\wedge\Box\rho S\,\rangle \\
&= \langle\,(\partial_\nu v)\Box\rho S\,\rangle \\
&= \langle\,(\partial_\nu v)\Box(\rho S)\,\rangle \\
&= \langle\,(\partial_\nu v)[\,(\Box\rho)S + \rho\Box S\,]\,\rangle \\
&= [\,(\partial_\nu v)\wedge\Box\rho)\,]\cdot S + \rho[\,(\partial_\nu v)\wedge\dot{\Box}\,]\cdot\dot{S}\,.
\end{aligned}
\tag{314}
$$

Now, we finally return to (305): [This equation will be needed soon.]

$$
\begin{aligned}
\partial^\mu N_\mu &= \gamma^\mu\big[\,S\cdot\partial_\mu(\frac{e}{m}Fe^{i\beta} - C) + (\partial_\nu v\wedge\gamma^\mu)\cdot W_\mu + m\partial_\mu\cos\beta\big] \\
&= \gamma^\mu\big[\,\partial_\mu S(\frac{e}{m}Fe^{i\beta} - C)\cdot\not{S} + \gamma^\mu(\partial_\mu v\wedge\gamma^\nu)\cdot W_\nu + m\gamma^\mu\partial_\mu\cos\beta\big]\,.
\end{aligned}
\tag{315}
$$

$$
\frac{\partial^\mu N_\mu}{-\rho} = \Box(\frac{e}{m}Fe^{i\beta} - C)\cdot\not{S} + \Box v\wedge\gamma^\nu\cdot\not{W}_\nu + m\Box\cos\beta\,.
\tag{316}
$$

So, now we're ready for (6.34):

$$
\begin{aligned}
\dot{p} &= eF\cdot v - \rho^{-1}\partial^\mu N_\mu \\
&= eF\cdot v + \frac{e}{m}\Box(Fe^{i\beta})\cdot\not{S} - \Box C\cdot\not{S} + \Box v\wedge\gamma^\nu\cdot\not{W}_\nu + m\Box\cos\beta\,.
\end{aligned}
\tag{317}
$$

According to the text, we substitute this last result into (2.34)

$$
\rho\dot{S} + \rho\,p\wedge v = \gamma_\mu\wedge N^\mu - \partial_\mu M^\mu\,,
\tag{2.34}
$$

and use (6.27) to get Eq. (6.36):

$$
\rho v\wedge p + \gamma_\mu\wedge N^\mu = \gamma_\mu\wedge T^\mu = \tfrac{1}{2}[\,v\wedge\gamma^\mu, W_\mu\,]\,.
\tag{6.36}
$$

7 Weyssenhoff Motion

There's not much for me to do in this short section, except to derive Eq. (7.5)

$$p = v(p \cdot v + \dot{S}) = (p \cdot v)v + v \cdot \dot{S}\,. \tag{7.5}$$

from Eq. (7.4).

$$\dot{S} = v \wedge p\,. \tag{7.4}$$

Okay,

$$\begin{aligned} v \cdot v \wedge p &= v \cdot \dot{S} \\ p &= v\, v \cdot p + v \cdot \dot{S} \\ &= (p \cdot v)v + v \cdot \dot{S} \\ &= v(p \cdot v + \dot{S})\,. \end{aligned} \tag{318}$$

since $v \wedge \dot{S} = 0$.

8 Interpretation of the Dirac Theory

No math in this section.

9 Appendix A: Matrix form of the Dirac Theory

We start off with γ_0 as a hermitian matrix and $\gamma_i (i = 1, 2, 3)$ anti-hermitian matrices. We will see both i and γ_5 used to represent $\gamma_0\gamma_1\gamma_2\gamma_3$. However, in what follows, I will use $i = \sqrt{-1}$ to be the complex unit imaginary.

We need to find a "spinor" u that satisfies both

$$\gamma_0 u = u \quad \text{and} \quad \gamma_2\gamma_1 u = iu\,, \tag{319}$$

where u is a 4×1 matrix. We have suitable matrix solutions for these equation in the assignments

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_2\gamma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{320}$$

The other Dirac matrices are shown on the next page.

Given a GA version of the ψ as

$$\psi = \rho^{1/2} e^{\frac{1}{2} i\beta} R\,, \tag{321}$$

(where R and thus ψ are even multivectors), we can arrive at the proper matrix version by the association

$$\Psi = \psi u\,, \tag{322}$$

which is Eq. (A3).

In the geometric algebra, we can write the Dirac Equation for an electron in an EM field as

$$\hbar \Box \psi \gamma_2\gamma_1 - eA\psi = m\psi\gamma_0\,, \tag{323}$$

which is Eq. (A4), and which corrects an error in the preprint version. Imagining this last equation as a matrix form and multiplying through on the right by u, we get

$$\hbar \Box \psi \gamma_2\gamma_1 u - eA\psi u = m\psi\gamma_0 u\,. \tag{324}$$

Using the equations in (319) for this:

$$\hbar i \Box \psi u - eA\psi u = m\psi u\,. \tag{325}$$

And finally,

$$\hbar i \Box \Psi - eA\Psi = m\Psi\,. \tag{326}$$

Or, alternatively,

$$(\hbar i \Box - eA)\Psi = m\Psi\,. \tag{327}$$

In coordinates, this becomes

$$\gamma^\mu (\hbar i \partial_\mu - eA_\mu)\Psi = m\Psi\,. \tag{328}$$

These last two equations constitute Eq. (A5) in the preprint.

The following is how to take the hermitian conjugate of an arbitrary gamma matrix γ_μ:

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \,, \tag{329}$$

which is Eq. (A6). More generally, let M be any multivector with real coefficients, then Eq. (A7) is

$$M^\dagger = \gamma_0 \widetilde{M} \gamma_0 \,, \tag{330}$$

where the tilde mean to take the reversion operator.

To obtain Eq. (A8), we need to know that $i \to \gamma_5$, $\widetilde{\gamma}_5 = \gamma_5$ and that $\tilde{\rho}^{1/2} = \rho^{1/2}$. Hence, from (321), we get

$$\widetilde{\psi} = \rho^{1/2} e^{\frac{1}{2} i\beta} \widetilde{R} \,, \tag{331}$$

since γ_5 commutes with all even multivectors. Thus,

$$\psi^\dagger = \gamma_0 \widetilde{\psi} \gamma_0 = \gamma_0 \rho^{1/2} e^{\frac{1}{2} i\beta} \widetilde{R} \gamma_0 = \rho^{1/2} e^{-\frac{1}{2} i\beta} \gamma_0 \widetilde{R} \gamma_0 = \rho^{1/2} e^{-\frac{1}{2} i\beta} R^\dagger \,, \tag{332}$$

which is Eq. (A9).

Before continuing, let's demonstrate the Dirac matrices with lower indices:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \,, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \,, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \,. \tag{333}$$

And, of course, there's also γ_5 (in the matrix algebra):

$$\gamma_5 \equiv -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \,. \tag{334}$$

And now, for combinations:

$$\tfrac{1}{2}(1 + \gamma_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \quad \tfrac{1}{2}(1 - i\gamma_2\gamma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{335}$$

Hence,

$$\tfrac{1}{4}(1 + \gamma_0)(1 - i\gamma_2\gamma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{336}$$

It seems like we have gone to a whole lot of trouble to get a matrix with a single nozero entry in it. But, as we shall see, this is a special matrix indeed, for

$$uu^\dagger = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}. \tag{337}$$

Thus

$$\begin{aligned} uu^\dagger &= \tfrac{1}{4}(1+\gamma_0)(1-i\gamma_2\gamma_1) \\ &= \tfrac{1}{4}(1+\gamma_0 - i\gamma_2\gamma_1 - i\gamma_0\gamma_2\gamma_1)\,, \end{aligned} \tag{338}$$

which is Eq. (A10), and a little more.

With $\Psi^\dagger = u^\dagger\psi^\dagger$, then we can get started on Eq. (A11).

$$\begin{aligned} \Psi\Psi^\dagger\gamma_0 &= \psi uu^\dagger\psi^\dagger\gamma_0 = \psi uu^\dagger\gamma_0^2\psi^\dagger\gamma_0 \quad (\text{since } \gamma_0^2 = 1) \\ &= \psi uu^\dagger\gamma_0(\gamma_0\psi^\dagger\gamma_0) = \psi u(u^\dagger\gamma_0)\widetilde{\psi} \\ &= \psi u(\gamma_0 u)^\dagger\widetilde{\psi} \\ &= \psi uu^\dagger\widetilde{\psi}\,. \end{aligned}$$

At this point, we make a substitution using (338) and that $\gamma_0\gamma_2\gamma_1 = \gamma_5\gamma_3$ to continue Eq. (A11):

$$\begin{aligned} \Psi\Psi^\dagger\gamma_0 &= \psi uu^\dagger\widetilde{\psi} \\ &= \tfrac{1}{4}\psi[\,1+\gamma_0 - i\gamma_2\gamma_1 + i\gamma_0\gamma_2\gamma_1\,]\widetilde{\psi} \\ &= \tfrac{1}{4}[\,\psi\widetilde{\psi} + \psi\gamma_0\widetilde{\psi} - i\psi\gamma_2\gamma_1\widetilde{\psi} - i\psi\gamma_5\gamma_3\widetilde{\psi}\,] \\ &= \tfrac{1}{4}\rho[\,e^{\beta\gamma_5} + v - ie^{\beta\gamma_5}\hat{s} + i\gamma_5\hat{s}\,]\,, \end{aligned} \tag{339}$$

where

$$\psi\widetilde{\psi} = \rho e^{\beta\gamma_5}\,, \tag{340}$$

and

$$\psi\gamma_0\widetilde{\psi} = \rho v\,, \tag{341}$$

and

$$\psi\gamma_2\gamma_1\widetilde{\psi} = \rho e^{i\beta}Ri\gamma_2\gamma_1\widetilde{R} = \rho e^{i\beta}\frac{2}{\hbar}\widehat{S}\,, \tag{342}$$

and

$$i\psi\gamma_5\gamma_3\widetilde{\psi} = i\gamma_5\psi\gamma_3\widetilde{\psi} = i\gamma_5\hat{s}\,. \tag{343}$$

For an arbitrary matrix M

$$\begin{aligned} \Psi^\dagger\gamma_0 M\Psi &= \mathrm{Tr}\,(\Psi\gamma_0 M\Psi^\dagger) = \mathrm{Tr}\,(M\Psi\Psi^\dagger\gamma_0) \\ &= \tfrac{1}{4}\{M[\,\psi\widetilde{\psi} + \psi\gamma_0\widetilde{\psi} - i\psi\gamma_2\gamma_1\widetilde{\psi} - i\psi\gamma_5\gamma_3\widetilde{\psi}\,]\} \\ &= \langle\, M\psi\widetilde{\psi}\,\rangle + \langle\, M\psi\gamma_0\widetilde{\psi}\,\rangle - i\langle\, M\psi\gamma_2\gamma_1\widetilde{\psi}\,\rangle - i\langle\, M\psi\gamma_0\gamma_2\gamma_1\widetilde{\psi}\,\rangle\,]\,, \end{aligned} \tag{344}$$

which is Eq. (A12). From the text: "The trace of a matrix in the Dirac matrix algebra is equal to four times the scalar part of the corresponding multivector in the space-time algebra..."

Let A, B, C, D be square matrices. Then, the trace of a product is invariant under cyclic permutation of the factors:

$$\mathrm{Tr}\,(ABCD) = \mathrm{Tr}\,(DABC) = \mathrm{Tr}\,(CDAB) = \mathrm{Tr}\,(BCDA)\,. \tag{345}$$

Hence,

$$\mathrm{Tr}\,(\Psi^\dagger \gamma_0 M \Psi) = \mathrm{Tr}\,(M\Psi\Psi^\dagger\gamma_0)\,. \tag{346}$$

Given

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}, \tag{347}$$

then

$$\gamma_0 M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ -m_{31} & -m_{32} & -m_{33} & -m_{34} \\ -m_{41} & -m_{42} & -m_{43} & -m_{44} \end{pmatrix}. \tag{348}$$

$$\begin{aligned} \Psi\Psi^\dagger\gamma_0\gamma_\mu &= \mathrm{Tr}\,(\gamma_\mu\Psi\Psi^\dagger\gamma_0) = \tfrac{1}{4}\mathrm{Tr}\,(\gamma_\mu\psi\gamma_0\widetilde{\psi}) \\ &= \langle\, \psi^\dagger\gamma_0\gamma_\mu\psi\,\rangle = \langle\, \gamma_\mu\psi\gamma_0\widetilde{\psi}\,\rangle \\ &= \rho\langle\, \gamma_\mu v\,\rangle = \rho\gamma_\mu\cdot v = \rho v_\mu\,. \end{aligned} \tag{349}$$

The Tetrode tensor in matrix form is Eq. (A15):

$$T_{\mu\nu} = \frac{i\hbar}{2}(\Psi^\dagger\gamma_0\gamma_\mu\partial_\nu\Psi - \partial_\nu\Psi^\dagger\gamma_0\gamma_\mu\Psi) - eA_\nu\Psi^\dagger\gamma_0\gamma_\mu\Psi\,, \tag{350}$$

which correctly adds a subscript to the vector A.

The first term of this last equation becomes

$$\begin{aligned} i\Psi^\dagger\gamma_0\gamma_\mu\partial_\nu\Psi &= i\mathrm{Tr}\,(\gamma_\mu(\partial_\nu\Psi)\Psi^\dagger\gamma_0) \\ &= \frac{i}{4}\mathrm{Tr}\,(\gamma_\mu\partial_\nu\psi[\,1+\gamma_0 - i\gamma_2\gamma_1 - i\gamma_5\gamma_3\,]\widetilde{\psi}) \\ &= i\langle\, \gamma_\mu\partial_\nu\psi\widetilde{\psi}\,\rangle + i\langle\, \gamma_\mu\partial_\nu\psi\gamma_0\widetilde{\psi}\,\rangle + \langle\, \gamma_\mu\partial_\nu\gamma_2\gamma_1\widetilde{\psi}\,\rangle + \langle\, \gamma_\mu\partial_\nu\psi i\gamma_5\gamma_3\widetilde{\psi}\,\rangle \\ &= i\langle\, \gamma_\mu\partial_\nu\psi\gamma_0\widetilde{\psi}\,\rangle + \langle\, \gamma_\mu\partial_\nu\psi i\gamma_5\gamma_3\widetilde{\psi}\,\rangle\,, \end{aligned} \tag{351}$$

which is Eq. (A16). The two terms that dropped out had only terms with an odd number of vector factors in their products, which can only produce vectors and trivectors in this algebra.

The second term of (350) becomes

$$\begin{aligned} i\partial_\nu\Psi^\dagger\gamma_0\gamma_\mu\Psi &= i\langle\, \gamma_\mu\psi\gamma_0\partial_\nu\widetilde{\psi}\,\rangle + \langle\, \gamma_\mu\psi\gamma_5\gamma_3\partial_\nu\widetilde{\psi}\,\rangle \\ &= i\langle\, \gamma_\mu\partial_\nu\psi\gamma_0\widetilde{\psi}\,\rangle - \langle\, \gamma_\mu\partial_\nu\psi\gamma_5\gamma_3\widetilde{\psi}\,\rangle\,. \end{aligned} \tag{352}$$

Putting this altogether into (350), we get Eq. (A18):

$$T_{\mu\nu} = \hbar\langle\, \gamma_\mu \partial_\nu \psi \gamma_5 \gamma_3 \psi \,\rangle - e\rho v_\mu A_\nu \,, \tag{353}$$

10 Appendix B: Divergence of J_μ

I repeat the error correction the preprint made concerning Ref. [1]: Equation (5) should be

$$\Box \cdot J_\mu = -2m \sin\beta\, e_3 \cdot J_\mu + 2ei\,(e_3 \wedge e_0 \wedge J_\mu \wedge A)\,. \tag{354}$$

So, now we derive this last equation by use of the Dirac equation. The Dirac equation (2.15) is given as

$$\hbar \Box \psi i \gamma_3 \gamma_0 = m\psi\gamma_0 + eA\psi\,. \tag{355}$$

By multiplying this on the right by $i\gamma_0\gamma_3\gamma_\mu\widetilde{\psi}$, we get

$$\hbar(\Box\psi) i\gamma_3\gamma_0 i\gamma_0\gamma_3\gamma_\mu\widetilde{\psi} = m\psi\gamma_0 i\gamma_0\gamma_3\gamma_\mu\widetilde{\psi} + eA\psi i\gamma_0\gamma_3\gamma_\mu\widetilde{\psi}\,. \tag{356}$$

Or,

$$\begin{aligned}\hbar(\Box\psi)\gamma_\mu\widetilde{\psi} &= -im\psi\gamma_3\gamma_\mu\widetilde{\psi} - ei A\psi\gamma_0\gamma_3\gamma_\mu\widetilde{\psi} \\ &= -im\rho e^{i\beta} R\gamma_3\gamma_\mu\widetilde{R} - e\rho i AR\gamma_0\gamma_3\gamma_\mu\widetilde{R} \\ &= -im\rho e^{i\beta} e_3 e_\mu - e\rho i A e_0 e_3 e_\mu\,. \end{aligned} \tag{357}$$

What about this ρ that I got that didn't appear in the preprint?

So, what is J_μ?

$$J_\mu \equiv \psi\gamma_\mu\widetilde{\psi}\,. \tag{358}$$

And we have the identity

$$\Box \cdot J_\mu = \langle\, \Box J_\mu \,\rangle\,. \tag{359}$$

Hence,

$$\begin{aligned}\Box \cdot J_\mu &= \langle\, \Box J_\mu \,\rangle = \langle\, \dot{\Box}\dot{\psi}\gamma_\mu\dot{\widetilde{\psi}} \,\rangle \\ &= \langle\, (\Box\psi)\gamma_\mu\widetilde{\psi} \,\rangle + \langle\, \dot{\Box}\psi\gamma_\mu\dot{\widetilde{\psi}} \,\rangle\,. \end{aligned} \tag{360}$$

We can transform the second term of this into a copy of the first term:

$$\begin{aligned}\langle\, \dot{\Box}\psi\gamma_\mu\dot{\widetilde{\psi}} \,\rangle &= \langle\, \gamma^\nu \dot{\partial}_\nu \psi\gamma_\mu\dot{\widetilde{\psi}} \,\rangle = \langle\, \gamma^\nu \psi\gamma_\mu\partial_\nu\widetilde{\psi} \,\rangle \\ &= \langle\, \gamma^\nu \psi\gamma_\mu\partial_\nu\widetilde{\psi} \,\rangle^{\sim} = \langle\, (\partial_\nu\psi)\gamma_\mu\widetilde{\psi}\gamma^\nu \,\rangle \\ &= \langle\, \gamma^\nu(\partial_\nu\psi)\gamma_\mu\widetilde{\psi} \,\rangle = \langle\, (\gamma^\nu\partial_\nu\psi)\gamma_\mu\widetilde{\psi} \,\rangle \\ &= \langle\, (\Box\psi)\gamma_\mu\widetilde{\psi} \,\rangle\,. \end{aligned} \tag{361}$$

Hence, (360) becomes

$$\Box \cdot J_\mu = 2\langle\, (\Box\psi)\gamma_\mu\widetilde{\psi}\,\rangle\,. \tag{362}$$

Now, we borrow from (357) and use that $e^{i\beta} = \cos\beta + i\sin\beta$ and that $e_\mu = J_\mu$ and separate out the two terms:

$$\begin{aligned}\Box \cdot J_\mu &= -\frac{2}{\hbar}\langle\, im\rho e^{i\beta}e_3e_\mu + e\rho Aie_0e_3e_\mu\,\rangle\\ &= -\frac{2}{\hbar}\langle\, im\rho(\cos\beta + i\sin\beta)e_3J_\mu\,\rangle - \frac{2e}{\hbar}\rho\langle\, Aie_0e_3J_\mu\,\rangle\\ &= \frac{2m}{\hbar}\rho\sin\beta\langle\, e_3J_\mu\,\rangle - \frac{2e}{\hbar}\rho\langle\, Aie_0e_3J_\mu\,\rangle\\ &= \frac{2m}{\hbar}\rho\sin\beta\, e_3\cdot J_\mu - \frac{2e}{\hbar}\rho\langle\, ie_0e_3J_\mu A\,\rangle\,. \end{aligned} \tag{363}$$

The selector of the last term of this can be rewritten as

$$\langle\, ie_0e_3J_\mu\,\rangle = \langle\, iA\wedge e_0\wedge e_3\wedge J_\mu\,\rangle = \langle\, ie_3\wedge e_0\wedge J_\mu\wedge A\,\rangle\,. \tag{364}$$

11 Conclusion

Well, it's nearly half a century since this Hestenes paper was published, and its had too little attention from the physics community in my opinion. Most of the time that Hestenes converted some standard form of math or physics into geometric algebra or geometric calculus, he'd get a cleaner formalism and often a widened domain of applicability. But for the Dirac equation, conversion is altogether different, for it entails a completely new paradigm in which the Dirac equation and the wave function are replaced by equations of conserved quantities for observables and their constituitive relations. It has also introduced a new parameter β, which is still somewhat mysterious.

I can see how that is a lot to accept for your typical conservative physics professor, who would rather play it safe. But I'm now retired and ready to sink my teeth into this subject.