More Vector Calculus Identities

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Abstract

Here we'll use either geometric calculus or Gibbs's vector calculus to prove some additional results in vector calculus.

1 Introduction

We'll assume that all our scalar functions have continuous first and second-order partial derivatives. In the identities to follow, assume that the bold variables are vectors and that ϕ, f, and g are scalar functions.

Warning! I've given names to three of these following identities. They are nonstandard and no one else uses them. Equations (1) – (9) were proved in the previous paper.

$$\nabla\cdot(\phi\mathbf{v}) = \mathbf{v}\cdot\nabla\phi + \phi\,\nabla\cdot\mathbf{v}\,, \tag{1}$$

$$\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi\nabla\times\mathbf{A}\,, \tag{2}$$

$$\nabla\times(\nabla\phi) = 0\,, \tag{3}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0\,, \tag{4}$$

$$\nabla\cdot(\nabla f\times\nabla g) = 0\,, \tag{5}$$

$$\nabla(\mathbf{A}\wedge\mathbf{B}) = \dot\nabla(\dot{\mathbf{A}}\wedge\mathbf{B}) + \dot\nabla(\mathbf{A}\wedge\dot{\mathbf{B}})\,, \tag{6}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})\,, \tag{7}$$

$$\nabla\left(\frac{1}{|\,\mathbf{x}-\mathbf{x}'\,|}\right) = -\nabla'\left(\frac{1}{|\,\mathbf{x}-\mathbf{x}'\,|}\right)\,, \tag{8}$$

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = -\mathbf{A}\cdot(\nabla\wedge\mathbf{B}) = -\mathbf{A}\cdot\nabla\mathbf{B} + \dot\nabla\mathbf{A}\cdot\dot{\mathbf{B}}\,. \tag{9}$$

Some new identities for us to look at are:

$$\nabla\cdot(\mathbf{A}\times\mathbf{C}) = (\nabla\times\mathbf{A})\cdot\mathbf{C} - (\nabla\times\mathbf{C})\cdot\mathbf{A}\,, \tag{10}$$

By re-ordering (7), we get

$$\nabla\times(\nabla\times\mathbf{B}) = \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B}\,, \tag{11}$$

Our first additional vector calculus identity is Eq. (10). To prove this, we need the following result

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = i\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\,, \tag{12}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors. I'll follow the convention that differential operators operate on everything to the right up to the end of the term it's in, unless it is restricted by delimitors. Thus,

$$\begin{aligned}
\nabla\cdot(\mathbf{A}\times\mathbf{C}) &= \dot\nabla\cdot(\dot{\mathbf{A}}\times\dot{\mathbf{C}}) \\
&= i\dot\nabla\wedge\dot{\mathbf{A}}\wedge\dot{\mathbf{C}} \\
&= i\dot\nabla\wedge\dot{\mathbf{A}}\wedge\mathbf{C}+i\dot\nabla\wedge\mathbf{A}\wedge\dot{\mathbf{C}} \\
&= i\mathbf{C}\wedge\dot\nabla\wedge\dot{\mathbf{A}}-i\mathbf{A}\wedge\dot\nabla\wedge\dot{\mathbf{C}} \\
&= \mathbf{C}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{C}) \\
&= (\nabla\times\mathbf{A})\cdot\mathbf{C}-(\nabla\times\mathbf{C})\cdot\mathbf{A}\,. \qquad (13)
\end{aligned}$$

Our second additional vector calculus identity is

$$(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{B}) = \nabla\cdot(\mathbf{A}\times(\nabla\times\mathbf{B}))+[\,\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}\,]\cdot\mathbf{A}\,, \qquad (14)$$

so let's prove it. This time, however, I won't use Geometric Calculus, because I can get it done with the identities above.

Okay, into (10) we'll replace $\mathbf{C}$ by $\nabla\times\mathbf{B}$:

$$\nabla\cdot(\mathbf{A}\times(\nabla\times\mathbf{B})) = (\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{B})-(\nabla\times(\nabla\times\mathbf{B}))\cdot\mathbf{A}\,. \qquad (15)$$

By using (11) in this last equation and re-ordering, we get

$$(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{B}) = \nabla\cdot(\mathbf{A}\times(\nabla\times\mathbf{B}))+[\,\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}\,]\cdot\mathbf{A}\,. \qquad (16)$$

So we're done, but let's take the time to look at a couple special cases. The first case is when $\mathbf{B}$ is divergenceless, then we have that

$$(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{B}) = \nabla\cdot(\mathbf{A}\times(\nabla\times\mathbf{B}))-\mathbf{A}\cdot(\nabla^2\mathbf{B})\,. \qquad (17)$$

The second special case is when $\mathbf{B}$ is curlless:

$$0 = [\,\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}\,]\cdot\mathbf{A}\,. \qquad (18)$$

And, since $\mathbf{A}$ is an arbitrary vector, we must have that

$$\nabla(\nabla\cdot\mathbf{B}) = \nabla^2\mathbf{B}\,, \qquad (19)$$

which is consistent with (11).

Then we have for our next identity

$$\nabla^2(\nabla\times\mathbf{A}) = \nabla\times(\nabla^2\mathbf{A})\,. \qquad (20)$$

A result I will need is:

$$\mathbf{a}\times\mathbf{b} = -i\,\mathbf{a}\wedge\mathbf{b}\,. \qquad (21)$$

Proof:

$$\begin{aligned}
\nabla^2(\nabla\times\mathbf{A}) &= \langle\,\nabla^2\nabla\times\mathbf{A}\,\rangle_1 \\
&= \langle\,\nabla^2(-i\nabla\wedge\mathbf{A})\,\rangle_1 \\
&= -\langle\,i\nabla^2(\nabla\wedge\mathbf{A})\,\rangle_1 \\
&= -\langle\,i\nabla\,\nabla\cdot(\nabla\wedge\mathbf{A})\,\rangle_1 \\
&= -\langle\,i\nabla\,(\nabla^2\mathbf{A}-\nabla\nabla\cdot\mathbf{A})\,\rangle_1 \\
&= -\langle\,i\nabla\,\nabla^2\mathbf{A}\,\rangle_1+\cancel{\langle\,i\nabla^2\nabla\cdot\mathbf{A})\,\rangle_1} \\
&= \nabla\times(\nabla^2\mathbf{A})\,. \qquad (22)
\end{aligned}$$

Done.