# Vector Calculus Identities

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**Abstract**

Here we'll use geometric calculus to prove a number of common Vector Calculus Identities. Unless stated otherwise, consider each vector identity to be in Euclidean 3-space. Most of the identities are recognizable in conventional form, but some are presented in geometric calculus form only.

## 1 Introduction

We'll assume that all our scalar functions have continuous first and second-order partial derivatives. In the identities to follow, assume that the bold variables are vectors and that $\phi$, $f$, and $g$ are scalar functions.

**Warning!** I've given names to three of these following identities. They are nonstandard and no one else uses them.

$$\nabla \cdot (\phi \mathbf{v}) = \mathbf{v} \cdot \nabla \phi + \phi \, \nabla \cdot \mathbf{v} \,, \tag{1}$$

$$\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,, \tag{2}$$

$$\nabla \times (\nabla \phi) = 0 \,, \tag{3}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \,, \tag{4}$$

$$\nabla \cdot (\nabla f \times \nabla g) = 0 \,, \tag{5}$$

$$\nabla (\mathbf{A} \wedge \mathbf{B}) = \dot{\nabla} (\dot{\mathbf{A}} \wedge \mathbf{B}) + \dot{\nabla} (\mathbf{A} \wedge \dot{\mathbf{B}}) \,, \tag{6}$$

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \,, \tag{7}$$

$$\nabla \left( \frac{1}{|\, \mathbf{x} - \mathbf{x}' \,|} \right) = -\nabla' \left( \frac{1}{|\, \mathbf{x} - \mathbf{x}' \,|} \right) , \tag{8}$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = -\mathbf{A} \cdot (\nabla \wedge \mathbf{B}) = -\mathbf{A} \cdot \nabla \mathbf{B} + \dot{\nabla} \mathbf{A} \cdot \dot{\mathbf{B}} \,. \tag{9}$$

$$\tag{10}$$

By the way, some of the proofs will use the identity

$$\nabla \wedge \nabla = 0 \,, \tag{11}$$

which we will not prove here.

Now to the longer identities:

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \,, \tag{12}$$

and

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \, (\nabla \cdot \mathbf{B}) - \mathbf{B} \, (\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \, \mathbf{A} - (\mathbf{A} \cdot \nabla) \, \mathbf{B} \,. \tag{13}$$

## 2 The Proof of Identity (1)

I refer to this identity as *Nickel's (dot) Identity*, but no one else does.

**Proof:** For arbitrary function $\phi$ and vector $\mathbf{v}$:

$$\begin{aligned}\nabla \cdot (\phi \mathbf{v}) &= \langle \nabla(\phi \mathbf{v}) \rangle \\ &= \langle (\nabla\phi)\mathbf{v} \rangle + \langle \phi \nabla \mathbf{v} \rangle \rangle \\ &= (\nabla\phi) \cdot \mathbf{v} + \phi \, \nabla \cdot \mathbf{v} \\ &= \mathbf{v} \cdot \nabla\phi + \phi \, \nabla \cdot \mathbf{v} \,. \end{aligned} \tag{14}$$

## 3 The Proof of Identity (2)

I refer to this identity as *Nickel's Cross Identity*, but, again, no one else does.

For $\phi$ and $\mathbf{A}$ arbitrary differentiable functions of $\mathbf{x}$,

$$\nabla \times (\phi \mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,. \tag{15}$$

**Proof:**

$$\begin{aligned}\nabla \times (\phi \mathbf{A}) &= -i \nabla \wedge (\phi \mathbf{A}) \\ &= -i \langle \nabla \phi \mathbf{A} \rangle_2 \\ &= -i \langle (\nabla\phi)\mathbf{A} + \phi \nabla \mathbf{A} \rangle_2 \\ &= -i [(\nabla\phi) \wedge \mathbf{A} + \phi \nabla \wedge \mathbf{A}] \\ &= (\nabla\phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,. \end{aligned} \tag{16}$$

## 4 The Proof of Identity (3)

**Proof:**

$$\begin{aligned}\nabla \times (\nabla\phi) &= -i \nabla \wedge (\nabla\phi) \\ &= -i \nabla \wedge \nabla\phi \\ &= 0 \,. \end{aligned} \tag{17}$$

## 5 The Proof of Identity (4)

**Proof:**

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{A}) &= \langle \nabla(\nabla \times \mathbf{A}) \rangle \\ &= \langle -i \nabla(\nabla \wedge \mathbf{A}) \rangle \\ &= -i \langle \nabla \wedge \nabla \wedge \mathbf{A} \rangle_3 \\ &= 0 \qquad (\text{since } \nabla \wedge \nabla = 0) \,. \end{aligned} \tag{18}$$

## 6 The Proof of Identity (5)

**Proof:**

$$\begin{aligned}\nabla \cdot (\nabla f \times \nabla g) &= \langle \nabla \cdot (-i\nabla f \wedge \nabla g)\rangle \\ &= \langle \nabla(-i\nabla f \wedge \nabla g)\rangle \\ &= -i\langle \nabla(\nabla f \wedge \nabla g)\rangle_3 \\ &= -i\langle (\nabla^2 f)\nabla g - (\nabla^2 g)\nabla f\rangle_3 \\ &= 0\,. \end{aligned} \tag{19}$$

## 7 The Proof of Identity (6)

**Proof:**

$$\begin{aligned}\nabla(\mathbf{A}\wedge\mathbf{B}) &= \nabla\tfrac{1}{2}(\mathbf{AB}-\mathbf{BA}) \\ &= \tfrac{1}{2}[\nabla(\mathbf{AB}) - \nabla(\mathbf{BA})] \\ &= \tfrac{1}{2}[\dot\nabla(\dot{\mathbf{A}}\mathbf{B}) + \dot\nabla(\mathbf{A}\dot{\mathbf{B}}) - \dot\nabla(\dot{\mathbf{B}}\mathbf{A}) - \dot\nabla(\mathbf{B}\dot{\mathbf{A}})] \\ &= \tfrac{1}{2}[\dot\nabla(\dot{\mathbf{A}}\mathbf{B}) - \dot\nabla(\mathbf{B}\dot{\mathbf{A}})] + \tfrac{1}{2}[\dot\nabla(\mathbf{A}\dot{\mathbf{B}}) - \dot\nabla(\dot{\mathbf{B}}\mathbf{A})] \\ &= \dot\nabla(\dot{\mathbf{A}}\wedge\mathbf{B}) + \dot\nabla(\mathbf{A}\wedge\dot{\mathbf{B}})\,. \end{aligned} \tag{20}$$

## 8 The Proof of Identity (7)

**Proof:**

Here we prove what I call the *Comstock Identity:*

$$\nabla^2 A = \nabla(\nabla\cdot A) - \nabla\times(\nabla\times A)\,, \tag{21}$$

by use of the *Klondike Identity:*[1]

$$\nabla A = \nabla\cdot A + \nabla\wedge A\,, \tag{22}$$

First, a couple identities:

$$\nabla\wedge\nabla = 0\,, \tag{23}$$

and

$$\mathbf{A}\wedge\mathbf{B} = i\mathbf{A}\times\mathbf{B}\,, \tag{24}$$

where $i$ is the pseudoscalar for 3-space, and the Comstock Identity is manifestly restricted to 3-space, whereas the Klondike Identity is not. In general, in (23), $A$ is a differentiable multivector function, but we will restrict it to being a differentiable vector in 3-space for this proof. A corollary to (23) is that

$$\nabla\nabla = \nabla\cdot\nabla \equiv \nabla^2\,. \tag{25}$$

Okay, first, we differentiate by $\nabla$ twice operating on $A$ and use the associative rule:

$$(\nabla\nabla)A = \nabla(\nabla A)\,. \tag{26}$$

Next, we use (22) on the RHS of (26), and use (25) on the LHS, to get

$$\nabla^2 A = \nabla(\nabla\cdot A + \nabla\wedge A)\,, \tag{27}$$

[1] This identity comes from Geometric Calculus, but nobody else calls it that.

which, after distributing the $\nabla$ operator, we get

$$\nabla^2 A = \nabla(\nabla \cdot A) + \nabla \cdot (\nabla \wedge A)\,. \tag{28}$$

I point out that this last equation is completely general. However, if we restrict $A$ to be an $n$-dimensional vector, then this equation makes a better fit to be called the 'Laplacian Vector Equation' than the Comstock Equation.

Anyway, our next task is to convert term $\nabla \cdot (\nabla \wedge A)$ in the 3-dimensional vector $A$ into $-\nabla \times\ (\nabla \times A)$.

$$\begin{aligned}\nabla \cdot (\nabla \wedge A) &= \nabla \cdot (i\nabla \times A) \\ &= \langle \nabla(i\nabla \times A)\rangle_1 \\ &= i\langle \nabla(\nabla \times A)\rangle_2 \\ &= i\nabla \wedge (\nabla \times A) \\ &= -\nabla \times (\nabla \times A)\,. \end{aligned} \tag{29}$$

Therefore, Eq. (28) becomes

$$\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)\,. \tag{30}$$

# 9 The Proof of Identity (8)

$$\nabla \left( \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \right) = -\nabla' \left( \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \right)\,, \tag{31}$$

where the prime means that the derivative operator is acting on the $\mathbf{x}'$ variable and ignoring the $\mathbf{x}$ variable. In other words,

$$\nabla \equiv \nabla_{\mathbf{x}}\,, \quad \text{and} \quad \nabla' \equiv \nabla_{\mathbf{x}'}\,. \tag{32}$$

**Proof:**

For (31) to be true, since it is a vector equation, it must hold for each component of the vector. Furthermore, since the computations for all three components are the same steps, we'll show this for just one component, say, the $x$ component, and other two follow similarly.

First, let's recall that

$$|\,\mathbf{x} - \mathbf{x}'\,| = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}\,. \tag{33}$$

Thus, we begin with the $\partial_x$:

$$\begin{aligned}\partial_x \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} &= \partial_x \frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}} \\ &= \left(-\frac{1}{2}\right) \frac{\partial_x (x-x')^2}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \\ &= \left(-\frac{1}{2}\right) \frac{2(x-x')(1)}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \\ &= -\frac{x-x'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}\,. \end{aligned} \tag{34}$$

Next, we have that

$$\begin{aligned}\partial_{x'}\frac{1}{|\,\mathbf{x}-\mathbf{x}'\,|} &= \partial_{x'}\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\\ &= \left(-\frac{1}{2}\right)\frac{\partial_{x'}(x-x')^2}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\\ &= \left(-\frac{1}{2}\right)\frac{2(x-x')(-1)}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\\ &= \frac{x-x'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\,. \end{aligned} \tag{35}$$

Therefore,

$$\partial_x\frac{1}{|\,\mathbf{x}-\mathbf{x}'\,|} = -\partial_{x'}\frac{1}{|\,\mathbf{x}-\mathbf{x}'\,|}\,. \tag{36}$$

Thus, from the preceding argument, we can conclude that (31) is true.

# 10 The Proof of Identity (9)

$$\begin{aligned}\mathbf{A}\times(\nabla\times\mathbf{B}) &= \langle -i\mathbf{A}\wedge(\nabla\times\mathbf{B}\rangle_1\\ &= -i\langle\mathbf{A}\wedge(\nabla\times\mathbf{B})\rangle_2\\ &= -i\langle\mathbf{A}(\nabla\times\mathbf{B})\rangle_2\\ &= -\langle\mathbf{A}(i\nabla\times\mathbf{B})\rangle_1\\ &= -\mathbf{A}\cdot(\nabla\wedge\mathbf{B})\\ &= -\mathbf{A}\cdot\nabla\mathbf{B}+\dot{\nabla}\mathbf{A}\cdot\dot{\mathbf{B}}\,. \end{aligned} \tag{37}$$

# 11 The Proof of Identity (12)

First, we take the last identity and write

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = -\mathbf{A}\cdot\nabla\mathbf{B}+\dot{\nabla}\mathbf{A}\cdot\dot{\mathbf{B}}\,. \tag{38}$$

And then we interchange vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{B}\times(\nabla\times\mathbf{A}) = -\mathbf{B}\cdot\nabla\mathbf{A}+\dot{\nabla}\mathbf{B}\cdot\dot{\mathbf{A}}\,. \tag{39}$$

Now, we add the last two equations, to get

$$\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) = -\mathbf{B}\cdot\nabla\mathbf{A}+\dot{\nabla}\mathbf{B}\cdot\dot{\mathbf{A}}-\mathbf{A}\cdot\nabla\mathbf{B}+\dot{\nabla}\mathbf{A}\cdot\dot{\mathbf{B}}\,. \tag{40}$$

The RHS of this can be simplified, given that $\dot{\nabla}\mathbf{B}\cdot\dot{\mathbf{A}} = \dot{\nabla}\dot{\mathbf{A}}\cdot\mathbf{B}$ and that

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \dot{\nabla}(\dot{\mathbf{A}}\cdot\mathbf{B})+\dot{\nabla}(\mathbf{A}\cdot\dot{\mathbf{B}})\,. \tag{41}$$

Therefore,

$$\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) = -\mathbf{B}\cdot\nabla\mathbf{A}-\mathbf{A}\cdot\nabla\mathbf{B}+\nabla(\mathbf{A}\cdot\mathbf{B})\,. \tag{42}$$

On solving for $\nabla(\mathbf{A}\cdot\mathbf{B})$, we have

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})\,. \tag{43}$$

## 12 The Proof of Identity (13)

$$
\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \langle -i\nabla \wedge (\mathbf{A} \times \mathbf{B} \rangle_1 \\
&= -i\langle \nabla \wedge (\mathbf{A} \times \mathbf{B})\rangle_2 \\
&= -i\langle \nabla(\mathbf{A} \times \mathbf{B})\rangle_2 \\
&= -\langle \nabla(i\mathbf{A} \times \mathbf{B})\rangle_1 \\
&= -\langle \nabla(\mathbf{A} \wedge \mathbf{B})\rangle_1 \\
&= -\dot{\nabla} \cdot (\dot{\mathbf{A}} \wedge \dot{\mathbf{B}}) \\
&= -(\dot{\nabla} \cdot \dot{\mathbf{A}})\dot{\mathbf{B}} + (\dot{\nabla} \cdot \dot{\mathbf{B}})\dot{\mathbf{A}} \\
&= -(\nabla \cdot \mathbf{A})\mathbf{B} - (\dot{\nabla} \cdot \mathbf{A})\dot{\mathbf{B}} + (\nabla \cdot \mathbf{B})\mathbf{A} + (\dot{\nabla} \cdot \mathbf{B})\dot{\mathbf{A}} \\
&= -(\nabla \cdot \mathbf{A})\mathbf{B} - \mathbf{A} \cdot \nabla\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{A} + \mathbf{B} \cdot \nabla\mathbf{A}\,. \qquad (44)
\end{aligned}
$$

On re-ordering this, we get

$$
\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}\,. \qquad (45)
$$