# Combinations are Integers

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February 25, 2020

**Abstract**

In this survey paper we visit some theorems that prove that the combinations of $n$ distinct things taken $k$ at a time, or $\binom{n}{k} = {}^nC_k$, is an integer.

## 1 Getting Started

If we take a good look at the fraction at the heart of the combinations of $n$ distinct things taken $k$ at a time, we see that

$$ {}^nC_k = \frac{n!}{k!(n-k)!} \,. \tag{1} $$

On the one hand $\binom{n}{k}$ is just a binomial coefficient and thus should count something. On the other hand, while I have no doubt that $\frac{n!}{k!}$ is an integer, as is also $\frac{n!}{(n-k)!}$, I think it strange that $\frac{n!}{k!(n-k)!}$ should be.

The following clever proof I found at math.stackexchange.com (see footnote for reference[1]). I will expand a bit on the proof given there.

## 2 Proof:

Let $S$ be a set of $n$ distinct elements and let $G_k$ be the set of all sequences that can be formed by taking $k$ distinct elements out of $S$. The order of $G_k$ is $|\, G_k \,| = {}^nP_k$, the number of premutations of $n$ things taken $k$ at a time, where

$$ {}^nP_k = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} \,. \tag{2} $$

Here's the clever part: Define a relation $R$ on the elements $a, b$ of $G_k$, such that $aRb$ (or, $a$ is related to $b$) if there exists a permutation of the 'components' of $a$ that transforms it into $b$.

As an example, let $S$ be the set of integers from 1 to 10, inclusive. The sequence $(1, 3, 7, 2)$ is related to $(7, 3, 1, 2)$ because there is a permutation on $a$ that permutes it into $b$.

[1] http://math.stackexchange.com/questions/11601/proof-that-a-combination-is-an-integer.

The relation $R$ is easily seen to be an equivalence relation (check this).[2] Two elements that are related to each other by an equivalence relation are said to be in the same *equivalence class*. If two sequences do not have the same elements as their components, then they are not related by $R$ and thus not in the same equivalence class.

So, every sequence in $G_k$ is in some equivalence class and all equivalence classes have the same order, namely, $k!$. These equivalence classes are mutually exclusive and collectively exhaustive. Let $m$ be the number of equivalence classes in $G_k$ by applying $R$. For our purposes, we don't care what $m$ is, except to know that it is a positive integer.

Let us label each equivalence classes by $A_i$ for some $i$. Therefore

$$G_k = \bigcup_{i=1}^{m} A_i \tag{3}$$

(which is a disjoint union), and thus

$$|\, G_k \,| = \sum_{i=1}^{m} |\, A_i \,| = m\, k!\,. \tag{4}$$

Therefore

$$k! \,\big|\, |\, G_k \,| \quad \text{or} \quad k! \,\big|\, [n!/(n-k)!]\,. \tag{5}$$

So, ${}^nC_k = \dfrac{n!}{k!(n-k)!}$ is an integer. Done.

[2]An equivalence relation must be reflexive, symmetric, and transitive.