Trigonometric Functions with Imaginary Arguments

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1 Introduction

In this short note we show how to deal with sines and cosine of imaginary arguments. In particular, we'll show that

$$\cos(ix) = \cosh x\,, \tag{1}$$
$$\sin(ix) = i\sinh x\,, \tag{2}$$
$$\cosh(ix) = \cos x\,, \tag{3}$$
$$\sinh(ix) = i\sin x\,, \tag{4}$$

where x is defined over the complex numbers. (The reader is assumed to know about complex exponentials and hyperbolic functions.)

2 Proofs

The proofs are fairly simple and they're all based on the following identities:

$$\cos(u) = \frac{e^{iu} + e^{-iu}}{2}\,, \tag{5}$$
$$\sin(u) = \frac{e^{iu} - e^{-iu}}{2i}\,, \tag{6}$$
$$\cosh(u) = \frac{e^{u} + e^{-u}}{2}\,, \tag{7}$$
$$\sinh(u) = \frac{e^{u} - e^{-u}}{2}\,. \tag{8}$$

► Proof of Equation (1):

In (5), let $u = ix$

$$\begin{aligned}\cos(ix) &= \frac{e^{-x} + e^{x}}{2}\\ &= \frac{e^{x} + e^{-x}}{2}\\ &= \cosh x\,.\end{aligned} \tag{9}$$

▶ Proof of Equation (2):

In (6), let $u = ix$

$$\begin{aligned} \sin(ix) &= \frac{e^{-x} - e^{x}}{2i} \\ &= -\frac{e^{x} - e^{-x}}{2i} \\ &= i \sinh x\,. \end{aligned} \tag{10}$$

▶ Proof of Equation (3):

In (7), let $u = ix$

$$\begin{aligned} \cosh(ix) &= \frac{e^{ix} + e^{-ix}}{2} \\ &= \cos x\,. \end{aligned} \tag{11}$$

▶ Proof of Equation (4):

In (8), let $u = ix$

$$\begin{aligned} \sinh(ix) &= \frac{e^{ix} - e^{-ix}}{2} \\ &= i\frac{e^{ix} - e^{-ix}}{2i} \\ &= i \sin x\,. \end{aligned} \tag{12}$$