Wirtinger Derivative of a Holomorphic Function

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Abstract

We will use the Cauchy-Riemann equations to show that the Wirtinger derivative of the second kind of a holomorphic function of a single complex variable is identically zero. The proof is easy.

1 Introduction

According to Wikipedia and other sources, the *Wirtinger derivatives* of *first* and *second* kinds of a function of a single complex variable, say z, are given, respectively, by[1]

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{1}$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2}$$

Now, let f be holomorphic on a domain $\mathcal{D}$. Then, show that

$$\frac{\partial f}{\partial \overline{z}} = 0\,. \tag{3}$$

2 Proof:

Since $f = f(z)$ is holomorphic, the Cauchy-Riemann equations apply over all points in $\mathcal{D}$. So, let $f(z) = u + iv$, where $z = x + iy$, and $u = u(x, y)$ is the real part of $f(z)$ and $v = v(x, y)$ is the imaginary part of $f(z)$. Then, the Cauchy-Riemann equations are given as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,. \tag{4}$$

Now, all we do is to substitute into $\dfrac{\partial f}{\partial \overline{z}}$ and expand (and employ the Cauchy-Riemann equations):

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\\ &= 0\,.\end{aligned} \tag{5}$$

[1] Actually, I added the adjectives " *first* and *second* kinds" for clarity. One can also say, "the Wirtinger derivative with respect to z or with respect to $\overline{z}$."