Solving First-Order Linear Differential Equations in 2 Variables

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Abstract

Solving first-order linear homogeneous differential equations in 2 variables will be done here with the assistance of some matrix theory. I assume that the matrix of coefficients is diagonalizable, which quickly leads to the solution.

1 Introduction:

I assume that the reader knows enough linear algebra to be able to solve for the eigenvalues and eigenvectors of a 2×2 matrix. Let's jump right in to see how we get such a matrix. Say we begin with the following couple of first-order differential equations in two variables x, y and independent variable time t.

$$\dot{x} = a_{11}x + a_{12}y\,, \tag{1a}$$

$$\dot{y} = a_{21}x + a_{22}y\,, \tag{1b}$$

where the coefficients are merely constants. Our next step is to form a vector differential equation out of this coupled equations:

$$\dot{X} = AX\,, \tag{2}$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix}\,, \tag{3}$$

and

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}\,, \tag{4}$$

and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\,. \tag{5}$$

Putting it all together, we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\,. \tag{6}$$

I might point out at this juncture that if the set of simultaneous equations to solve is an $n \times n$ coefficient matrix, the method of solution is the same, though the amount of calculation goes up accordingly.

Now, if matrix A has a complete set of eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, then we can form the matrix P, such that

$$P = [\,\mathbf{v}_1, \mathbf{v}_2\,] = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} . \tag{7}$$

So, what does a diagonal matrix look like? It looks like this:

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} , \tag{8}$$

where d_1 and d_2 are just numbers.

Okay, say we have this matrix P, how do we use it? To say that matrix A is diagonalizable by matrix P, we mean

$$D = P^{-1}AP\,. \tag{9}$$

This fact comes out of the theory of linear algebra. I won't prove it; I'll just use it. Now, in order to diagonalize the matrix A in Eq. (2), we'll need to make a change of variables. To that end, let's introduce variables Y_1 and Y_2 as the components of the 2×1 matrix Y by

$$Y = P^{-1}X\,, \tag{10}$$

where

$$Y \equiv \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} . \tag{11}$$

Now, the components of matrix P, and thus of matrix P^{-1} as well, are not functions of time, hence we can write

$$\dot{Y} = P^{-1}\dot{X}\,. \tag{12}$$

We're almost there. Let's insert PP^{-1} into (2) to get

$$\dot{X} = APP^{-1}X\,. \tag{13}$$

Then we multiply through on the left by P^{-1}, to get

$$P^{-1}\dot{X} = P^{-1}APP^{-1}X\,. \tag{14}$$

Lastly, applying Eqs. (9), (10), and (12) to this last equation, we have that

$$\dot{Y} = DY\,, \tag{15}$$

which in matrix form looks like

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} . \tag{16}$$

On deconstructing this, we get the simple uncoupled system of equations:

$$\dot{Y}_1 = d_1 Y_1 , \tag{17}$$

$$\dot{Y}_2 = d_2 Y_2 , \tag{18}$$

whose solutions are

$$Y_1 = c_1 e^{d_1 t} , \tag{19}$$

$$Y_2 = c_2 e^{d_2 t} , \tag{20}$$

where c_1 and c_2 are as yet arbitrary constants, which are usually determined by provided initial conditions on the time derivatives.

But we want the answer in terms of x and y. For these, we solve (10) for X:

$$X = PY , \tag{21}$$

which in matrix form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} c_1 e^{d_1 t} \\ c_2 e^{d_2 t} \end{bmatrix} , \tag{22}$$

or

$$x = v_{11} c_1 e^{d_1 t} + v_{21} c_2 e^{d_2 t} , \tag{23a}$$

$$y = v_{12} c_1 e^{d_1 t} + v_{22} c_2 e^{d_2 t} . \tag{23b}$$

2 Practice Problem

Solve the following problem by use of the method shown above.

$$\dot{x} = 2x - y , \tag{24a}$$

$$\dot{y} = -x + 2y . \tag{24b}$$

So, next we convert to matrix form:

$$X = \begin{bmatrix} x \\ y \end{bmatrix} , \tag{25}$$

and

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} , \tag{26}$$

Using (6),

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} . \tag{27}$$

Putting it all together, we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} . \tag{28}$$

The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 1$. The corresponding eigenvectors[1] are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} , \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{29}$$

Thus,

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} . \tag{30}$$

and

$$P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} . \tag{31}$$

Using these values in (9), we get that

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} . \tag{32}$$

Next, using (15) we have that

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} , \tag{33}$$

with solutions:

$$Y_1 = c_1 e^{3t} , \tag{34a}$$

$$Y_2 = c_2 e^{t} . \tag{34b}$$

Solving for the variables x, y inside of X, we have from (21) that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{t} \end{bmatrix} . \tag{35}$$

So, on multiplying them out, we get

$$x = c_1 e^{3t} + c_2 e^{t} , \tag{36}$$

$$y = -c_1 e^{3t} + c_2 e^{t} . \tag{37}$$

[1] What's important about eigenvectors is their directions, not their lengths. So, feel free to rescale them for your own convenience, but then hold them fixed.