First-order Differential Equations of Riccati-type

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Abstract

In this paper I demonstrate a well-known solution to the simple Riccati differential equation. Knowledge of first-order linear nonhomogeneous differential equations is assumed.

In the theory of differential equations, the general nonlinear equation of Riccati type looks like:

$$y' = a(x)y^n + b(x)y + c(x)\,, \tag{1}$$

where n is an integer greater than or equal to 2. In this paper, I will solve the simpler form, where $n = 2$.

$$y' = a(x)y^2 + b(x)y + c(x)\,. \tag{2}$$

We begin by making the following nonintuitive transformation ansatz

$$y = y_0 + \frac{1}{W}\,, \tag{3}$$

where, of course, $W \neq 0$ on the relevant domain. Substituting the ansatz into (2), we get

$$y_0' - \frac{W'}{W^2} = a(x)\big[y_0^2 + 2\frac{y_0}{W} + \frac{1}{W^2}\big] + b(x)\big[y_0 + \frac{1}{W}\big] + c(x)\,. \tag{4}$$

On multiplying through by W^2, we get

$$y_0'W^2 - W' = a(x)[y_0^2W^2 + 2y_0W + 1] + b(x)[y_0W^2 + W] + c(x)W^2\,. \tag{5}$$

Now, at first sight, all we've done is to replace the y^2 term with a W^2 term, but something wonderful is about to happen. Let's collect all the W^2 terms together:

$$[y_0' - a(x)y_0^2 - b(x)y_0 - c(x)]W^2 - W' = a(x)[2y_0W + 1] + b(x)W\,. \tag{6}$$

Now, if y_0 happens to be a known solution to (2), often referred to as the 'particular' solution to it, then the quadratic term on the LHS drops out, leaving us with

$$W' + [2a(x)y_0 + b(x)]W = -a(x)\,. \tag{7}$$

This last equation is a first-order linear nonhomogeneous differential equation of the form

$$Y' + p(x)Y = q(x) \tag{8}$$

with known formula

$$Y(x) = \frac{\int u(x)q(x)\,dx + c}{u(x)} \tag{9}$$

where c is a constant of integration and

$$u(x) = \exp\Big(\int p(x)\,dx\Big)\,. \tag{10}$$

So, solving $W(x)$ in (7), with

$$q(x) = -a(x) \quad \text{and} \quad p(x) = 2a(x)y_0 + b(x) \tag{11}$$

then

$$W(x) = \frac{-\int u(x)a(x)\,dx + c}{u(x)}\,, \tag{12}$$

where c is a constant of integration and

$$u(x) = \exp\Big(\int [2a(x)y_0 + b(x)]\,dx\Big)\,. \tag{13}$$

Hence, the full solution to (3) is

$$y = y_0 + \frac{u(x)}{-\displaystyle\int u(x)a(x)\,dx + c}\,. \tag{14}$$

One last comment: The general solution to this Riccati-type equation is founded on one's ability to find a particular solution to it, but I have found no general methodology to do so. However, when the Riccati equation is simple, such as when $a(x)$, $b(x)$, $c(x)$ are all constant, then one may be able to successfully guess at it.